DOE/ER/03077-276

Courant Mathematics and
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U.S. Department of Energy

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J. Jones

Research and Development Report

Supported by the Applied Mathematical Sciences
program of the Office of Energy Research,
. Department of Energy under Contract
DE-AC02-76ER03077

thematics and Computers

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ABSTRACT

An expanding cylindrically or spherically symmetric detonation is analyzed in a regime in which the radius of the detonation is much greater than the width of the reaction zone. Under this assumption the fundamental equations may be approximated by a system of autonomous ordinary differential equations for the flow velocity and a reaction progress variable. The independent variable of this system is a radial variable in the rest frame of the detonation front. The radius of curvature and the detonation speed enter the system as parameters. At zero curvature this system reduces to the plane wave equations of Zeldovich, von Neumann and Doering. The plane wave equations possess a degenerate bifurcation point with a nilpotent linear part, which bifurcates into a saddle node when the radius is finite. Any smooth transonic solution must pass through the saddle node. This fact determines the wave speed implicitly as a function of radius. To leading order, the correction to the detonation speed as a function of curvature is proportional to the curvature, on the basis of formal and numerical considerations.

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Table of Contents

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1. Supported in part by National Science Foundation Grant Nos. MCS-82-07965 and MCS-83-01255.
2. Supported in part by Army Research Office, Contract No. DAAG-29-84-K0130.
3. Supported in part by U.S. Dept. of Energy, Contract No. DE-AC02-76ER03077.

1. Introduction. The diverging detonation is interesting because the nonlinear interaction between the chemistry and the hydrodynamics is displayed in a particularly sharp form. The theoretical study of detonations in a diverging geometry began in the late 1940's with the work of Eyring, Powell, Duffey, and Parlin [9] and of H. Jones [12]. These papers were largely concerned with the relationship between the detonation velocity and the finite diameter of the charge which has come to be called the "diameter effect". Wood and Kirkwood (1954) [19] were the first to elucidate the relationship between the wave velocity and the radius of curvature of the detonation wave. This fundamental work has been extended by a number of authors [3, 7, 8, 15]. In all of these works the geometry considered was a cylindrically symmetric charge with the detonation propagating parallel to the axis of symmetry. An excellent review of this problem can be found in Fickett and Davis [10], pp. 199 - 229. The axial geometry is useful to both experimentalists and theorists because steady state solutions exist. For the experimentalist this means that stationary diverging detonations can be produced in the laboratory. For the theorist this reduces the problem to an analysis of a system of ordinary differential equations.

Although a substantial body of work on steady state, diverging detonations has accumulated, there has been little progress in developing an understanding of the dynamics of diverging detonations. One reason for this is the presence of nonuniformities that appear in straightforward attempts to perform an asymptotic analysis of this problem. In the present work, we present an analysis of a spherically symmetric detonation, or a radially propagating cylindrically symmetric detonation (independent of the axial

coordinate). Although this is not a steady state problem, we will show that to first order in the ratio of reaction zone width to radius of curvature, the detonation may be modeled by a system of autonomous ordinary differential equations in space, with time entering as a parameter. The nonuniformities are exhibited as unboundedness of the vector field along the sonic locus, at which the flow velocity relative to the shock equals the sound speed. A bounded transonic solution, i.e. a solution crossing the sonic locus, must pass through a critical point of the vector field. We will employ the methods of bifurcation theory to show that an appropriate critical point exists. The evolution in time of the detonation is then determined by a shooting problem connecting the state at the shock interface to the critical point. Our main result, derived in conjunction with the numerical results of Bukiet [5, 6], is a demonstration on the level of numerical computation that the leading order correction to the detonation wave velocity is proportional to the curvature. A second main result is the formal derivation of ordinary differential equations for the leading order large radius asymptotics for the internal structure of the expanding detonation wave. A third main result is the analysis of this system of ordinary differential equations. We show that the plane wave sonic critical point bifurcates into a unique critical point which is a saddle point, and that this critical point is a smooth function of the shock curvature. A normal form is proposed for the bifurcation point, and a one parameter unfolding is presented illustrating the effect of curvature on the phase portrait.

In a recent work, Bdzil and Stewart [4] have independently presented a related asymptotic theory for the dynamics of curvilinear detonations. They

employ the strong shock approximation and a specialized rate law for which the reaction terminates at a finite distance behind the shock, and coincides with the sonic transition. These idealizations permit them to study two dimensional detonation dynamics and to deal with the characteristic nonuniformities in a more traditional fashion. Although Bdzil and Stewart emphasize that their analysis cannot be expected to apply to more general reaction kinetics, it is interesting to note that they also find that the leading order correction to the speed of the detonation wave is linear in the shock curvature.

2. Derivation of the Model. The equations for a cylindrically symmetric, transport free, reactive, polytropic gas are

$$\rho_t + m_r + \frac{m}{r} = 0 \tag{2.1}$$

$$m_t + \left(\frac{m^2}{\rho} + p\right)_r + \frac{m^2}{\rho r} = 0$$

$$E_t + \left(\frac{m(E+p)}{\rho}\right)_r + \frac{m(E+p)}{\rho r} = 0$$

$$(\rho\lambda)_t + \left(\frac{(\rho\lambda)m}{\rho}\right)_r + \left(\frac{(\rho\lambda)m}{\rho r} - \rho R\right) = 0.$$

Here $m = \rho u$ is the mass flux, ρ is the density, u is the radial velocity and p is the pressure, t is the time and r is the radial coordinate. λ is the reaction progress parameter, which varies from 0 (all reactant) to 1 (all product). The total energy density is $E = \rho e + m^2/2\rho$, where the specific internal energy e

is given by

$$(2.2) \qquad e = \frac{p}{\rho(\gamma - 1)} + (1 - \lambda)q \, ,$$

γ is the polytropic gas constant and q is the heat released per unit mass by the complete reaction. R is the reaction rate function, which we assume has the Arrhenius form

$$R(\lambda, T) = \begin{cases} k(1 - \lambda)\exp\left(-\dfrac{A}{T}\right) & , T_c \leq T \\ 0 & , T < T_c \end{cases}$$

where $T = \dfrac{p}{\rho}$ is the temperature (using units in which the ideal gas constant is unity), k is the rate multiplier, and A is the activation energy. The constant T_c is the critical temperature below which the reaction rate is taken to be identically zero. The role of T_c is to avoid the famous "cold boundary problem". Without such a cutoff temperature there are no solutions for large times because all of the reactant is consumed upstream of the shock. To obtain the equations of a spherically symmetric detonation, simply include a factor of 2 in the source term of the density equation, i.e. replace $\dfrac{m}{r}$ by $\dfrac{2m}{r}$.

The internal energy may be eliminated by substituting (2.2) into (2.1) to obtain an equation for the pressure. Choosing ρ, u, p, and λ as dependent variables then leads to the system

$$(2.3) \qquad \rho_t + \mathrm{u}\rho_r + \mathrm{u}_r\rho + \frac{\rho \mathrm{u}}{r} = 0$$

$$\rho \mathrm{u}_t + \rho \mathrm{u}\mathrm{u}_r + p_r = 0$$

$$p_t + \mathrm{u} p_r + \gamma p \mathrm{u}_r + \frac{\gamma p \mathrm{u}}{r} = q(\gamma - 1)\rho R(\lambda, T)$$

$$\lambda_t + \mathrm{u}\lambda_r = R(\lambda, T) \ .$$

We desire an asymptotic solution for an outgoing detonation which has been propagating for a long time, so that initialization transients have died out. For such a solution the radius of the detonation must be much greater than the other relevant physical length scale, the reaction zone width. The reaction zone width w may be defined as the distance from the shock front to the point at which some fixed fraction f of the reactant has been consumed. Denote the radius of the shock as a function of time by $z(t)$. The assumptions on t and z are then

$$w << z \quad \text{and} \quad \frac{w}{\dot{z}} << t$$

where $\dot{z} \equiv \dfrac{dz}{dt}$ is the wave speed. In the $t \to \infty$ limit the solution approaches a plane detonation (or plane shock if the detonation fails). Thus the wave speed $\dot{z}$ is expected to approach the constant plane wave value. It is well known that the speed of the plane detonation is determined by the initial and final Hugoniot curves alone and is independent of the reaction rate R and therefore independent of the reaction zone width. This means that z and t are linearly related as $t \to \infty$, independently of w.

The reaction zone width for a steady plane detonation may be defined explicitly by integrating the rate equation

$$w_0 = \int_0^f \frac{\dot{z} - \mathrm{u}}{R} \, d\lambda \, ,$$

where $f \in (0, 1)$. The relevant observation here is that w scales as $\frac{1}{R}$. Let ϵ be the ratio of the reaction zone width w_0 to a typical radius. Since the wave speed is constant in the plane wave limit, z and t are nearly proportional for sufficiently large t. If we assume that w_0 and the plane wave speed $\dot{z}_0$ are $O(1)$, then z and t are $O\left(\frac{1}{\epsilon}\right)$. We now transform the differential equations (2.3) to an appropriate form for asymptotic analysis by translating the radial variable r to the rest frame of the shock and rescaling t and z to be $O(1)$ for large times. Define

(2.4)
$$\tau \equiv \epsilon t$$

$$\zeta(\tau) \equiv \epsilon z(t)$$

$$x \equiv z(t) - r = \frac{\zeta(\tau)}{\epsilon} - r$$

$$\mathrm{v} \equiv \dot{\zeta} - \mathrm{u} = \dot{z} - \mathrm{u}$$

where $\dot{\zeta} \equiv \frac{d\zeta}{d\tau}$. Note that the transformation $r \to x$ involves a reversal of orientation, as shown in Fig. 2.1. This convention was chosen so that x will be positive behind the shock in the reaction zone. The flow velocity relative to the shock, oriented parallel to x, is v. Now change variables from t, r, z, and u to τ, x, ζ, and v respectively to obtain

(2.5)
$$\epsilon\rho_\tau - \mathrm{v}\rho_x + \mathrm{v}_x\rho = \frac{-\epsilon\rho(\dot{\zeta} - \mathrm{v})}{\zeta - \epsilon x}$$

$$\epsilon \mathrm{v}_\tau + \mathrm{v}\mathrm{v}_x + \frac{p_x}{\rho} = \epsilon\ddot{\zeta}$$

$$\epsilon p_\tau + \mathrm{v}p_x + \gamma p \mathrm{v}_x = q(\gamma - 1)\rho R(\lambda, T) - \frac{\epsilon\gamma p(\dot{\zeta} - \mathrm{v})}{\zeta - \epsilon x}$$

$$\epsilon\lambda_\tau + v\lambda_x = R(\lambda,T).$$

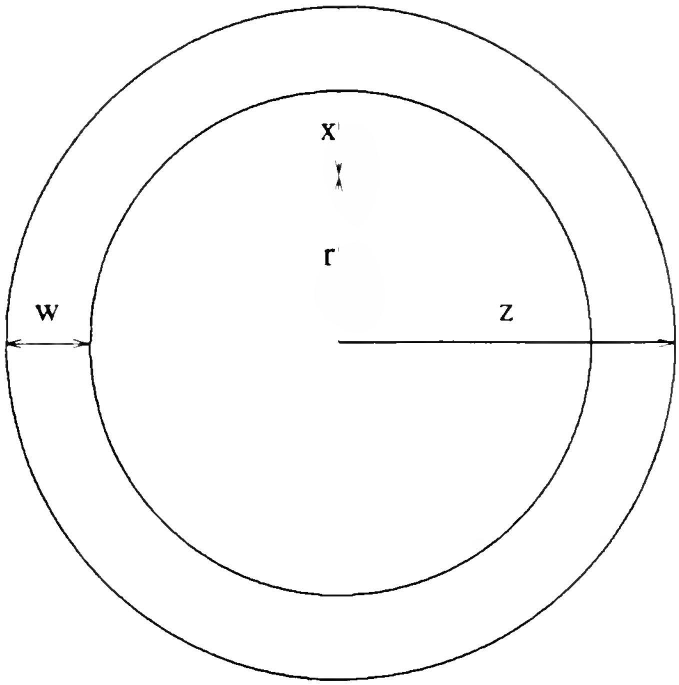

Figure 2.1

There are two alternate derivations of (2.5) which are worth mentioning. First, one could transform to dimensionless variables and set all $O(1)$ parameters to 1. Alternately, rather than taking w and x to be $O(1)$ and z and t large, one could assume that z and t are $O(1)$ and that w and x are small by rescaling R and doing an "inner expansion" in x behind the shock.

The system (2.5) has the form

(2.5a) $$\epsilon \mathbf{w}_\tau + \mathbf{O}(\mathbf{w}) \cdot \mathbf{w}_x = \mathbf{h}(\mathbf{w}, x, \epsilon).$$

It will be seen in Section 3 that the desired solution of (2.5) must possess both subsonic and supersonic regions with a smooth transition between. At such a transition point the quasilinear operator **O** becomes singular. This singularity may be interpreted as a turning point of the system. If we attempt to solve for $\mathbf{w}_x$ in (2.5a) near the turning point we find that the x derivatives blow up unless certain solvability conditions are satisfied. Specifically $\mathbf{h} - \epsilon \mathbf{w}_T$ must lie in the range of **O**. We will not attempt here to prove asymptotic convergence of solutions of (2.5) as $\epsilon \to 0$ to the planar ($\epsilon = 0$) solution, but will proceed in the spirit of formal perturbation theory by assuming on physical grounds that a smooth transonic solution of (2.5) exists which is differentiable with respect to ϵ. The turning point structure will be apparent in the final version of the model which we will derive in Section 3. We thus assume that a representation

$$\rho(x, t, \epsilon) = \rho_0(x, t) + \epsilon\rho_1(x, t) + \rho_{rem}(x, t, \epsilon)$$

exists in ϵ for $\rho(x, t, \epsilon)$, where

$$\lim_{\epsilon \to 0} \frac{\rho_{rem}(x, t, \epsilon)}{\epsilon\rho_1(x, t)} = 0$$

uniformly in x and t. We also assume analogous representations for v, p, λ and ζ. Now expand (2.5) formally in powers of ϵ, grouping together all terms of each power. The zeroth order equations are

(2.6)
$$\mathrm{v}_0\rho_{0x} + \mathrm{v}_{0x}\rho = 0$$

$$\mathrm{v}_0\mathrm{v}_{0x} + \frac{p_{0x}}{\rho_0} = 0$$

$$\mathrm{v}_0 p_{0x} + \gamma p_0 \mathrm{v}_{0x} = q(\gamma - 1)\rho_0 R_0$$

$$v_0 \lambda_{0x} = R_0.$$

These are the equations for a one dimensional steady state detonation studied by Zeldovich, von Neumann and Doering (ZND). The hydrodynamic equations may be integrated to obtain ρ, v, and p as functions of λ. The rate equation then constitutes an ordinary differential equation for λ as a function of x,

(2.7) $$\rho_0 v_0 = m \ (= \text{constant})$$

$$\frac{p_0 - p_s}{V_0 - V_s} = -m^2$$

$$2\mu^2 q \lambda_0 = (V_s - \mu^2 V_0) p_s - (V_0 - \mu^2 V_s) p_0$$

$$\frac{d\lambda}{dx} = \frac{R_0}{v_0},$$

where $\mu^2 = \dfrac{\gamma - 1}{\gamma + 1}$, and $V \equiv \dfrac{1}{\rho}$ is the specific volume. The s subscript indicate variables evaluated immediately behind the shock. The first of equations (2.7) states that the mass flux is constant in the shock frame. The second equation defines a line of slope $-m^2$ in the p, V plane, and is referred to as the *Rayleigh line*. The third of equations (2.7) defines a family of *Hugoniot curves* in the p, V plane. The intersections of the Rayleigh line with the $\lambda = 0$ Hugoniot curve determine the possible shock transitions. The solution terminates at point where the Rayleigh line intersects the $\lambda = 1$ Hugoniot curve. In general, there is a one parameter family of solutions, parameterized by the mass flux m, or for a given ambient state ahead of the

shock, by the wave speed $\dot{\zeta}_0$.

The first order equations are

(2.8) $$\rho_{0\tau} + v_0\rho_{1x} + v_1\rho_{0x} + \rho_0 v_{1x} + \rho_1 v_{0x} = -\frac{\rho_0(\dot{\zeta}_0 - v_0)}{\zeta_0}$$

$$\rho_0 v_{0\tau} + \rho_0 v_0 v_{1x} + \rho_0 v_1 v_{0x} + \rho_1 v_0 v_{0x} + p_{1x} = \rho_0 \ddot{\zeta}_0$$

$$p_{0\tau} + v_0 p_{1x} + v_1 p_{0x} + \gamma p_0 v_{1x} + \gamma p_1 v_{0x}$$

$$= q(\gamma - 1)(\rho_0 R_1 + \rho_1 R_0) - \frac{\gamma p_0(\dot{\zeta}_0 - v_0)}{\zeta_0}$$

$$\lambda_{0\tau} + v_0\lambda_{1x} + v_1\lambda_{0x} = R_1 \ .$$

Since the zeroth order equations are steady state, the time derivatives drop out of the first order equations, and $\ddot{\zeta} = 0$. Thus the first order solution is "quasi-steady" in the sense that the solution is given by steady state equations with time entering as a parameter (through ζ_0 and $\dot{\zeta}_0$). Note also that the ϵx term which appears in the denominator of the geometric source terms in (2.5) does not appear through first order. Thus, in the reaction zone, the deviation of the flow divergence from the value at the shock is a second order effect. If the time derivatives and ϵx terms are dropped from (2.5) one obtains a system of autonomous ordinary differential equations

(2.9) $$v\rho_x + v_x\rho = \frac{-\epsilon\rho(\dot{\zeta} - v)}{\zeta}$$

$$vv_x + \frac{p_x}{\rho} = 0$$

$$vp_\tau + \gamma p v_\tau = q(\gamma - 1)\rho R(\lambda,T) - \frac{\epsilon\gamma p(\dot{\zeta} - v)}{\zeta}$$

$$v\lambda_\tau = R(\lambda,T) \ .$$

It is easily verified that (2.9) possesses the same formal zeroth and first order equations as does the original system (2.5). On the basis of the assumed regularity of the asymptotic expansion for (2.5) we now take (2.9) as our first order model for the expanding detonation and turn our attention to an analysis of this system. Of course one could also pursue the more usual course of studying the linear non-autonomous perturbation equations (2.8), but the autonomous system (2.9) permits a much cleaner treatment by the technique of phase space analysis.

3. Reduction of Order for the System (2.9). Note in (2.9) that ϵ and ζ occur only in the ratio $\frac{\epsilon}{\zeta}$ so that ϵ is really a redundant parameter. This redundancy may be removed by setting $\epsilon = 1$. This inverts the scale transformation, so that $\tau = t$ and $\zeta = z$. These identifications will be made henceforth.

The steady state energy equation is

$$e_\tau + pV_\tau = 0 \ .$$

The velocity equation in (2.9) can be written as

$$\frac{1}{2}(v^2)_\tau + Vp_\tau = 0 \ .$$

These two equations may then be added to obtain

$$\left(\frac{1}{2}\mathrm{v}^2 + e + Vp\right)_\tau = 0,$$

which integrates to yield Bernoulli's Law:

(3.1) $$\frac{1}{2}\mathrm{v}^2 + e + Vp = f(t)\ .$$

Denote values upstream of the shock by the subscript a (for "ambient"). It will be assumed that the ambient state is constant and unreacted ($\lambda_a = 0$), and that the ambient flow velocity is zero ($\mathrm{u}_a = 0$, or $\mathrm{v}_a = \dot{z}$). For our polytropic equation of state, (3.1) becomes

(3.2) $$\frac{1}{2}\mathrm{v}^2 + \frac{1}{\gamma - 1}c^2 - \lambda q = \frac{1}{2}\dot{z}^2 + \frac{c_a^2}{\gamma - 1}\ .$$

Here $c = (\gamma p/\rho)^{\frac{1}{2}}$ is the sound speed. Note that we are able to connect across the shock to the ambient state since Bernoulli's law is one of the Rankine-Hugoniot jump conditions. This fact determines the function $f(t)$ in (3.1).

Now eliminate p_τ between the velocity and pressure equations to obtain

(3.3a) $$\mathrm{v}_\tau = \frac{q(\gamma - 1)R - \dfrac{(\dot{z} - \mathrm{v})c^2}{z}}{c^2 - \mathrm{v}^2}$$

$$= \frac{q(\gamma - 1)k(1 - \lambda)\exp\left(-\dfrac{A\gamma}{c^2}\right) - \dfrac{(\dot{z} - \mathrm{v})c^2}{z}}{c^2 - \mathrm{v}^2}.$$

By (3.2) c^2 is a known function of v and λ. Therefore the right hand side of the equation above is also a known function of v and λ. This form of the

velocity equation may be combined with the rate equation

$$(3.3b) \qquad \lambda_\tau = \frac{R}{\mathrm{v}}$$

to obtain a self contained system of two equations for u and λ.

For a steady undriven plane detonation, the reaction zone terminates at the Chapman-Jouguet, or CJ point $\mathrm{v} = c$ in the Hugoniot diagram. At this point the Rayleigh line is tangent to the $\lambda = 1$ Hugoniot curve. This criterium determines a unique solution of (2.7). The diverging detonation is weakened by rarefactions produced behind the shock front and terminates below the sonic point on the weak detonation branch of the $\lambda = 1$ Hugoniot curve. This means that a sonic transition must occur in the reaction zone from the subsonic flow behind the shock to the supersonic flow at termination. However, the denominator in the velocity equation (3.3a) vanishes at a sonic transition, so for a smooth sonic transition to occur, the numerator must vanish simultaneously. The sonic transition is just the turning point mentioned in Section 2, and the condition that the numerator vanish is equivalent through first order to the solvability condition for $\mathbf{w}_\tau$ in (2.5a). We may use substitute v^2 for c^2 in (3.2) to obtain the *sonic locus*

$$(3.4a) \qquad \mathrm{v}^2 = \frac{2c_a^2}{\gamma + 1} + \mu^2\dot{z}^2 + 2\mu^2 q\lambda .$$

A solution of (3.3) which crosses the sonic locus will be called *transonic*. We thus seek a transonic solu..ion of (3.3) which satisfies

$$(3.4b) \qquad q(\gamma - 1)R - \frac{1}{\dot{z}}c^2(\dot{z} - \mathrm{v}) = 0$$

at the sonic transition. A solution $(\mathrm{v}_c, \lambda_c)$ of the system (3.4) will be

referred to as a *sonic critical point*. Note that when $z = \infty$, equation (3.4b) yields $R = 0$ at the sonic critical point. If the detonation has not failed this implies $\lambda = 1$ at the sonic critical point, consistent with the aforementioned result that a steady undriven plane detonation terminates at the CJ point.

The wave speed $\dot{z}_0$ and final ($\lambda = 1$) flow velocity v_{CJ} for the plane wave may be determined from equations (2.7) and (3.4a) and the condition $v_{CJ} = c_{CJ}$, where c_{CJ} is the sound speed at the CJ point. The result is

$$(3.5) \qquad \dot{z}_0 = \left[(\gamma^2 - 1)qc_a^{-2} + 1 + (((\gamma^2 - 1)qc_a^{-2} + 1)^2 - 1)^{1/2}\right]^{1/2} c_a$$

$$v_{CJ} = \left(\frac{2c_a^2}{\gamma + 1} + \mu^2\dot{z}_0^2 + 2\mu^2 q\right)^{1/2}.$$

The positive solution in (3.5) must be chosen in order to satisfy the jump conditions at the shock.

The system will (3.3) be easier to analyze if transformed into a more conventional form. Define the singular change of variable

$$(3.6) \qquad y \equiv \int^{x} \frac{\exp\left(-\dfrac{A\gamma}{c(x')^2}\right)dx'}{(c(x')^2 - v(x')^2)v(x')}.$$

Now change variables from x to y to obtain

$$(3.7) \qquad v_y = q(\gamma - 1)k(1 - \lambda)v - \frac{(\dot{z} - v)v}{z}c^2 \exp\left(\frac{A\gamma}{c^2}\right)$$

$$\lambda_y = k(1 - \lambda)(c^2 - v^2).$$

Observe that the structure of the phase curves in the (v, λ) plane is unaltered by this change of independent variable since the transformed vector

field is proportional to the initial vector field. The integral curves have simply been reparameterized to eliminate the singularity in the denominator of the velocity equation. The right hand side of (3.7) is now bounded in the region of interest, and the sonic critical point defined by (3.4) is recognized as a stationary point of the system.

Several observations about (3.7) can be made immediately. Since the first equation is proportional to v, the λ axis is a phase curve of the system. Likewise, the $\lambda = 1$ line is a phase curve, since the second equation has a factor of $1 - \lambda$. These two phase curves intersect in a fixed critical point (0, 1). When $z = \infty$, the vector field is proportional to $1 - \lambda$, so that the entire $\lambda = 1$ line is stationary. The $z = \infty$ sonic critical point is thus embedded in a manifold of critical points. We will see that the $z = \infty$ sonic critical point is a *bifurcation point* for the system, i.e. a point in the phase plane where the topology of the phase curves is unstable to small perturbations of the vector field. These notions will be made more precise in the following section.

For a fixed ambient state ahead of the shock, the Rankine-Hugoniot jump conditions determine a one parameter family of solutions behind the shock, parameterized by the wave speed $\dot{z}$. The critical point equations (3.4) define v_c and λ_c as functions of z and $\dot{z}$. The desired solution of (3.7) will connect the ambient state (via the jump conditions) to a sonic critical point; this *shooting problem* defines the functional relationship between z and $\dot{z}$.

The method of determining the full dynamic solution of (3.7) is now clear. We first analyze the system allowing z and $\dot{z}$ to vary as independent parameters, thus determining the phase plane structure throughout some

domain in parameter space. We then may recover the leading order dynamics by solving the shooting problem. This paper is primarily concerned with the first half of this procedure, i.e. determining the phase plane structure of (3.7). The shooting problem has been solved numerically by Bukiet [5, 6].

After eliminating c_a between (3.2) and the second of equations (3.5), we obtain

$$c^2 - v^2 = -q(\gamma - 1)(1 - \lambda) - \frac{\gamma + 1}{2}(v^2 - v_b^2), \tag{3.8}$$

where

$$v_b \equiv (v_{CJ}^2 + \mu^2(\dot{z}^2 - \dot{z}_0^2))^{1/2}$$

is the flow velocity at the $z = \infty$ sonic bifurcation point. With z and $\dot{z}$ independent, v_b is now a function of $\dot{z}$ and $\dot{z}_0$.

We may obtain a single differential equation for $v(\lambda)$ by dividing the first equation in (3.7) by the second to obtain

$$(1 - \lambda)(c^2 - v^2)\frac{dv}{d\lambda} = q(\gamma - 1)(1 - \lambda)v - \frac{(\dot{z} - v)v}{zk} c^2 \exp\left(\frac{A\gamma}{c^2}\right). \tag{3.9}$$

This equation clearly shows the (nonlinear) turning point character of the equation. When $z = \infty$, we may use (3.5) and (3.8) to write equation (3.9) in the form

$$\left(1 - \frac{v_b^2 - 2q\mu^2(1 - \lambda)}{v^2}\right)dv + \frac{2q\mu^2}{v}d\lambda = 0 .$$

The left hand side of this equation is the differential of the function

$$f(v, \lambda) = v + \frac{v_b^2 - 2q\mu^2(1 - \lambda)}{v} + C .$$

Denoting by (v_r, λ_r) any fixed reference point, we have

$$v + \frac{v_b^2 - 2q\mu^2(1 - \lambda)}{v} = v_r + \frac{v_b^2 - 2q\mu^2(1 - \lambda_r)}{v_r} \tag{3.10}$$

The choice $v_r = v_b$, $\lambda = 1$ yields the separatrix solution

$$(v - v_b)^2 + 2q\mu^2(\lambda - 1) = 0 \tag{3.11}$$

for the plane wave. This result could also have been obtained by solving (2.7) for $v(\lambda)$.

Denote by $v = g(\lambda, z, \dot{z})$ the solution of the shooting problem for (3.9) connecting the ambient state to a sonic critical point. Evaluating g at the critical point, one obtains an implicit ordinary differential equation

$$v_c(z, \dot{z}) = g(\lambda_c(z, \dot{z}), z, \dot{z})$$

for the radius $z(t)$ and the wave speed $\dot{z}(t)$.

4. Statement of Main Results. The main result of this paper, taken in conjunction with Bukiet [5, 6] is that the velocity of the expanding detonation is equal to the plane wave velocity plus a correction which is to lowest order linear in the shock curvature $\kappa \equiv 1/z$. Moreover the constant of proportionality is determined by the separatrix solution of (3.7). This statement follows on a numerical level from our equations together with their numerical solution and the comparison to the numerical solution of the full Euler equations by Bukiet. One consequence of this result is that standard methods of computation of detonation waves [13] which use the experimental values of the planar detonation velocity can be improved in accuracy by these

corrections. Moreover since the correction can be computed from the chemistry, we believe that the correction can be predicted from some phenomenological equation of state and rate law, at least after the latter have been recalibrated to reflect the new requirement that they reproduce both planar speeds and leading order curvature corrections. Such a predictive capability would minimize the amount of experimental calibration necessary to use this new theory in numerical computations.

We begin our analysis of the model (3.7) with the statement of our main theorem. We will see in subsequent sections that the linear part of the vector field at the plane wave sonic critical point is singular, and is consequently a bifurcation point for the system. Theorem 4.1 identifies the effect of curvature in (3.7) as a perturbation of the sonic bifurcation point.

THEOREM 4.1 Assume that there is a radius z_1 such that $T_c < T$ whenever $0 < x$ and $z_1 < z$. Then there is a $\kappa_{max} > 0$ and a neighborhood $(\dot{z}_{min}, \dot{z}_{max})$ of $\dot{z}_0$ such that

i) The sonic bifurcation point $v_b \equiv (v_{CJ}^2 + \mu^2(\dot{z}^2 - \dot{z}_0^2))^{1/2}$, $\lambda_b = 1$ in the vector field (3.7) bifurcates into a saddle point as $\kappa \equiv \dfrac{1}{z}$ is increased from zero, for all $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$.

ii) For κ_{max} sufficiently small, and $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$, the saddle point in i) is the unique sonic critical point of (3.7).

iii) The location of the saddle point in the phase plane is a C^∞ function of $\kappa \in (0, \kappa_{max})$ and $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$.

iv) The restriction of the vector field (3.3) to the stable separatrix of the saddle point is continuous, uniformly in $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$.

v) The unique smooth transonic solution of (3.3) is given by the stable separatrix of the saddle point.

In [3, 5] Bukiet solved the shooting problem numerically for the functional dependence of the wave speed $\dot{z}$ on the curvature κ. The separatrix solution in Theorem 4.1 is computed simultaneously, employing the wave speed determined by the shooting problem. The computations by Bukiet indicate that the wave speed depends linearly on the curvature to leading order.

We now present some of the tools which this analysis will require. Those familiar with bifurcation theory for vector fields may wish to skip to Section 5.

By the *phase flow* $\boldsymbol{\phi}_y^{\mathbf{f}}(\mathbf{w})$ of a vector field $\dfrac{d\mathbf{w}}{dy} = \mathbf{f}(\mathbf{w})$ we mean the general solution of the differential equation interpreted as a map taking an initial point $\mathbf{w}$ at $y = 0$ to its image at some later value of y. Thus

$$\frac{d}{dy}\boldsymbol{\phi}_y^{\mathbf{f}}(\mathbf{w}) = \mathbf{f}(\mathbf{w}) \text{ and } \boldsymbol{\phi}_0^{\mathbf{f}}(\mathbf{w}) = \mathbf{w}.$$

DEFINITION 4.2 Two vector fields $\dfrac{d\mathbf{w}}{dy} = \mathbf{f}(\mathbf{w})$ and $\dfrac{d\mathbf{w}}{dy} = \mathbf{g}(\mathbf{w})$ are *C^j equivalent* if there exists a C^j bijection $\mathbf{h}(\mathbf{w})$ such that

$$\mathbf{h}(\boldsymbol{\phi}_y^{\mathbf{f}}(\mathbf{w})) = \boldsymbol{\phi}_y^{\mathbf{g}}(\mathbf{h}(\mathbf{w})).$$

The function $\mathbf{h}$ maps the phase flow of $\mathbf{f}$ onto the phase flow of $\mathbf{g}$ preserving orientation and parametrization by y. C^0 equivalence is also referred to as *topological equivalence*.

Let $K \subset \mathrm{R}^n$ be closed with non-empty interior and with ∂K smooth. We equip the space of C^j vector fields on K with the norm

$$\|\mathbf{f}\|_{j,\,K} \equiv \max_{|i| \leq j} \quad \sup_{\mathbf{w} \in K} \left| \frac{\partial^{|i|}}{\partial^{i}\mathbf{w}} \mathbf{f}(\mathbf{w}) \right| ,$$

where i denotes the multiindex $(i_1, \cdots, i_n)$, $|i| \equiv i_1 + \cdots + i_n \leq j$, and $\partial^i \mathbf{w} \equiv \partial^{i_1} \mathbf{w}_1 \cdots \partial^{i_n} \mathbf{w}_n$. The norm $\| \ \|_{j,\,K}$ provides a precise measure of the smallness of a C^j perturbation of a vector field.

DEFINITION 4.3 Let $\dfrac{d\mathbf{w}}{dy} = \mathbf{f}(\mathbf{w})$ be a C^j vector field in R^n. The vector field $\mathbf{g}(\mathbf{w})$ is called a C^j *perturbation of size* ϵ of $\mathbf{f}(\mathbf{w})$ if there is a compact set K in R^n such that $\mathbf{f} = \mathbf{g}$ on $\mathrm{R}^n - K$, and $\|(\mathbf{f} - \mathbf{g})\|_{j,\,K} < \epsilon$.

The set of C^j perturbations of size ϵ of $\mathbf{f}$ is just a ball of radius ϵ centered at $\mathbf{f}$ in the normed linear space of C^j vector fields.

DEFINITION 4.4 A vector field $\mathbf{f}$ is said to be *structurally stable* if there is an $\epsilon > 0$ such that all C^1 perturbations of $\mathbf{f}$ of size ϵ are C^0 equivalent to $\mathbf{f}$.

Structural stability is a precise form of the notion of robustness of a model. Many of the qualitative features of a structurally stable system are preserved under perturbations of the system.

We will also need local versions of the above definitions.

DEFINITION 4.5 Two vector fields $\dfrac{d\mathbf{w}}{dy} = \mathbf{f}(\mathbf{w})$ and $\dfrac{d\mathbf{w}}{dy} = \mathbf{g}(\mathbf{w})$ defined respectively on open sets Ω_f and Ω_g in R^n are said to be C^j *equivalent at* $\mathbf{w}_0$ if there exists a point $\mathbf{w}_0 \in \Omega_f \cap \Omega_g$, a C^j homeomorphism $\mathbf{h}(\mathbf{w})$ and a neighborhood $\Omega' \subset \Omega_f$ of $\mathbf{w}_0$ such that $\mathbf{h}(\mathbf{w}_0) = \mathbf{w}_0$ and $\mathbf{h}(\boldsymbol{\phi}_y^{\mathbf{f}}(\mathbf{w})) = \boldsymbol{\phi}_y^{\mathbf{g}}(\mathbf{h}(\mathbf{w}))$ for all $\mathbf{w} \in \Omega'$.

More generally, local equivalence need not be required to preserve a point $\mathbf{w}_0$. However, the most common application of local equivalence is in a neighborhood of a critical point. Any equivalence must map a critical point onto a critical point. If the critical point is unique, then its image under an equivalence map is completely determined. In this case the more general type of equivalence may be factored into a translation and a point preserving equivalence as in Definition 4.5. Similar comments may be made concerning Definitions 4.6 and 4.7.

Definitions 4.2 and 4.5 are clearly symmetric with respect to interchange of $\mathbf{f}$ and $\mathbf{g}$. They are also transitive and reflexive (the identity map takes $\mathbf{f}$ onto itself).

DEFINITION 4.6 Let $\dfrac{d\mathbf{w}}{dy} = \mathbf{f}(\mathbf{w})$ be a C^j vector field in a neighborhood Ω of $\mathbf{w}_0$ in R^n. A vector field $\mathbf{g}(\mathbf{w})$ is called a *C^j perturbation of size ϵ of* $\mathbf{f}$ *at* $\mathbf{w}_0$ if $\mathbf{g}(\mathbf{w}_0) = \mathbf{f}(\mathbf{w}_0)$ and if there is a neighborhood $\Omega' \subset \Omega$ of $\mathbf{w}_0$ in which we have $\|(\mathbf{f} - \mathbf{g})\|_{j,\,\bar{\Omega}'} < \epsilon$.

The C^j perturbations with which we will be primarily concerned are those obtained by adding to $\mathbf{f}$ the remainder of a Taylor series expansion. Denote by $\mathcal{P}_{m,j}$ the class of real analytic functions on $U \subset R^m$ which are $O(|\mathbf{w}|^j)$ for $\mathbf{w} \in U$, where U is a neighborhood of the origin. If $\mathbf{p}(\mathbf{w})$ is a vector field on R^m, and each if of the components $\mathbf{p}_i$, $i = 1, 2, \cdots, m$ are elements of $\mathcal{P}_{m,j+1}$ defined on neighborhoods U_i of the origin, then the field $\mathbf{g} = \mathbf{f} + \mathbf{p}$ is a C^j perturbation of $\mathbf{f}$ in a neighborhood $U = \bigcap\limits_{1 \le i \le n} U_i$ of the origin. Since the partial derivatives of $\mathbf{p}$ vanish at the origin through order j, the size of the perturbation $\mathbf{f} + \mathbf{p}$ may be made arbitrarily small by restricting

the size of U.

DEFINITION 4.7 A vector field $\mathbf{f}$ is said to be *structurally stable at* $\mathbf{w}_0$ if there is an $\epsilon > 0$ such that all C^1 perturbations of size ϵ of $\mathbf{f}$ at $\mathbf{w}_0$ are C^0 equivalent to $\mathbf{f}$ at $\mathbf{w}_0$. By *bifurcation point* we mean any point at which $\mathbf{f}$ is not structurally stable.

In general, *bifurcation* refers to a change in the topological equivalence class of the vector field under a perturbation. In practice, the perturbation is controlled by a set of *bifurcation parameters*, and the different equivalence classes for the perturbed vector field correspond to different regions of the parameter space. An n parameter family of perturbations of a bifurcation point is called an *n parameter unfolding* of the bifurcation. We will show in Section 6 that κ is a bifurcation parameter for the plane wave sonic bifurcation point, so that (3.7) is a one parameter unfolding of the bifurcation. Local equivalence is usually defined by means of the elegant machinery of jets [1]. We have chosen a more prosaic definition which is adequate for our purposes. We will also make use of the following restricted form of structural stability.

DEFINITION 4.8 Let K denote a subset of the set of all C^1 perturbations at $\mathbf{w}_0$. A vector field $\mathbf{f}$ is said to be C^k *stable to class K perturbations at* $\mathbf{w}_0$ if all perturbations in K of $\mathbf{f}$ at $\mathbf{w}_0$ are C^k equivalent to $\mathbf{f}$ at $\mathbf{w}_0$.

PROPOSITION 4.9 Let $\mathbf{f}(\mathbf{w})$ be a continuous vector field in a domain of R^n with no critical points. Then $\mathbf{f}$ is locally structurally stable.

PROPOSITION 4.10 (Hartman-Grobman Theorem). Let $\mathbf{w}_c$ be an isolated critical point of a smooth vector field $\mathbf{f}(\mathbf{w})$. If none of the eigenvalues

of the Jacobian derivative $D\mathbf{f}(\mathbf{w}_c)$ of $\mathbf{f}$ at at $\mathbf{w}_0$ have a zero real part, then $\mathbf{f}$ is topologically equivalent to the linear system $\frac{d\mathbf{w}}{dx} = D\mathbf{f}(\mathbf{w}_c)\cdot(\mathbf{w} - \mathbf{w}_c)$ at $\mathbf{w}_c$.

Proposition 4.9 follows easily from the Rectification Theorem for continuous vector fields, which states that the vector field is C^1 equivalent to a constant field in a neighborhood of a noncritical point. (The Rectification Theorem may be found, for example, in Arnold [2].) Any sufficiently small perturbation of a constant field is noncritical and therefore rectifiable, but all nonzero constant fields are topologically equivalent. This implies that a bifurcation point is necessarily a critical point. A proof of Proposition 4.10 is presented in Arnold [1]. It is easily verified that the linear system in Proposition 4.10 is structurally stable at the critical point, and thus the Hartman-Grobman theorem implies that a bifurcation point must possess at least one eigenvalue with zero real part. Critical points possessing no eigenvalues with a zero real part are termed *hyperbolic*.

The Hartman-Grobman theorem proves structural stability for hyperbolic critical points by relating them to the critical points of a topologically equivalent linear system which is easy to analyze. The method we will employ to analyse the sonic bifurcation point is analogous to what the Hartman-Grobman theorem accomplishes for hyperbolic critical points, except that nonlinear terms are required because of the degeneracy of the bifurcation point. We transform the vector field in a neighborhood of the bifurcation point by a smooth change of variables to a field which is topologically equivalent but simpler in form. It turns out that in most cases, the topological equivalence class of a vector field at a critical point contains

polynomial fields. In such a case there are fields of smallest degree, and possessing the fewest number of terms. The study of these minimal fields was initiated by Poincaré, and the fields are referred to as *normal forms* for the equivalence class. To find such a field, we will take advantage of the analyticity of the vector field by expanding in powers of the deviations $\hat{v}$, $\hat{\lambda}$ from the critical values $(v_b, 1)$. We then seek a diffeomorphism of the phase space that preserves the linear part of the vector field while eliminating as many nonlinear terms as possible. This procedure is customarily carried out order by order; first quadratic terms are eliminated, then cubic terms and so on. In the present case, only the second degree terms are required. The next step is to show that the transformed field may be truncated at some finite order to obtain a topologically equivalent field. This latter step is usually the hardest, in part because it is usually necessary to identify the class of perturbations which preserve the equivalence class, and in part because the construction of topological equivalence maps is often a highly nontrivial enterprise. The normal form thus possesses the same local topological structure as the original field, but is much easier to study. This method may also be applied to an unfolding of the bifurcation, so we may speak of a normal form for the unfolding. We will demonstrate the first step of this technique in Section 6. As indicated in the discussion following Definition 4.5, the second step of local topological equivalence is factorized, and the subproblem of translation to a fixed critical point is also solved in Section 6. Excellent introductions to the theory of normal forms for vector fields are available in Arnold [1] and in Guckenheimer and Holmes [11].

Assume that $\mathbf{A}$ is the matrix of the linear part of a two dimensional analytic vector field $\mathbf{f}(\mathbf{w})$ with a critical point at the origin, so that $\mathbf{f}$ has the form

$$\frac{d\mathbf{w}}{dy} = \mathbf{f}(\mathbf{w}) = \mathbf{A}\cdot\mathbf{w} - \mathbf{f}^{(2)}(\mathbf{w}) \ ,$$

where $\mathbf{f}^{(2)} = O(|\mathbf{w}|^2)$. For each integer $n \geq 2$, the linear operator $\mathbf{A}$ induces a linear operator $\mathbf{L_A}$ on the linear space $\mathcal{J}_{2,n}$ of 2-vectors having entries which are homogeneous nth degree polynomials in $\mathbf{w}_1$, $\mathbf{w}_2$,

$$(4.1) \qquad (\mathbf{L_A}\cdot\mathbf{h})_i \equiv \sum_{j=1}^{2} \frac{\partial \mathbf{h}_i}{\partial \mathbf{w}_j}(\mathbf{A}\cdot\mathbf{w})_j - (\mathbf{A}\cdot\mathbf{h})_i .$$

Note that $\mathbf{L_A}\cdot\mathbf{h}$ is just the Lie bracket $[\mathbf{h}, \mathbf{A}\cdot\mathbf{w}]$ of $\mathbf{h}$ with $\mathbf{A}\cdot\mathbf{w}$. If $\mathbf{L_A}$ were nonsingular, all nth degree terms in the Taylor series for $\mathbf{f}$ could be eliminated by the nonlinear change of variables $\overline{\mathbf{w}} = \mathbf{w} + \mathbf{L_A}^{-1}\cdot\mathbf{h}$ where $\mathbf{h}$ denotes the vector of nth degree terms in the series. In general, only those nth degree terms of $\mathbf{f}$ which lie in the range of $\mathbf{L_A}$ can be eliminated. Those elements of $\mathcal{J}_{2,n}$ which do not lie in the range of $\mathbf{L_A}$ cannot be eliminated by a smooth change of variables and are termed *resonant*. The resonant terms of $\mathbf{f}$ contain the essential nonlinear contributions to the phase plane structure. Applying $\mathbf{L_A}$ to the standard basis $(\mathbf{w}_1^j\mathbf{w}_2^l, 0)^T$ and $(0, \mathbf{w}_1^j\mathbf{w}_2^l)^T$, $j + l = n$, yields a set of vectors which span the range of $\mathbf{L_A}$. A basis for the range may be chosen from this set. We need never include more than codim(range($\mathbf{L_A}$)) in the set of resonant vectors. This is because two resonant vectors which differ only by an element of range($\mathbf{L_A}$) are smoothly equivalent (the element of range($\mathbf{L_A}$) may be transformed away). Consequently we may identify the set of resonant vectors with the non-zero vectors of the quotient space $\mathcal{J}_{2,n}$/range($\mathbf{L_A}$). Choose any basis for this quotient space. These basis

vectors are equivalence classes of elements of $J_{2,n}$. Now choose any particular representatives for these equivalence classes. These representatives define a maximal set of resonant vectors, i.e. **f** may be smoothly transformed into a vector field with second degree terms consisting of a linear combination of this maximal set of resonant vectors. In practice this maximal set is chosen from among the standard basis vectors, if possible, in order to produce the maximum simplification of the vector field. Extensions of these ideas to higher dimensional vector fields and their associated spaces $J_{m,n}$ of homogeneous vector valued polynomials are obvious.

In Sections 5 and 6 we begin a local analysis of the bifurcation point using the ideas presented above. In Section 5 we present and study a one parameter unfolding of a proposed normal form for the plane wave sonic critical point point $v = c$, $\lambda = 1$. This degenerate critical point possesses two zero eigenvalues and is the primary bifurcation point for the system. The unfolding parameter represents the effect of shock curvature on the phase plane structure. For this simplified vector field the plane wave sonic critical point bifurcates into a unique saddle point as the shock curvature $1/z$ is increased from zero. In Section 6 we show that the unique saddle structure observed in Section 5 is preserved under a class of perturbations which includes (3.7), and that the critical point itself is a smooth function of $1/z$.

5. The Proposed Normal Form. We propose

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \nu \begin{pmatrix} -\alpha\hat{\lambda} \\ (\hat{\lambda} - \eta\kappa)\hat{v} \end{pmatrix} \tag{5.1}$$

as a normal form for the vector field (3.7) at the bifurcation point. Here $\kappa \equiv \frac{1}{z}$ is the bifurcation parameter. The coefficients ν, α, and η are positive. The variables $\hat{v}$ and $\hat{\lambda}$ here denote v and λ translated to the transonic critical point, which remains fixed at the origin as κ is varied. In this section we investigate the properties of this proposed normal form. The results here will assist in understanding the properties of the transonic critical point, as well as lay a foundation for an eventual proof of local topological equivalence with (3.7) at the critical point.

If the first of the equations (5.1) is divided by the second, a separable ordinary differential equation

$$\hat{v}d\hat{v} = -\frac{\alpha\hat{\lambda}}{\hat{\lambda} - \eta\kappa}d\hat{\lambda}$$

is obtained for the phase curves $\hat{v}(\hat{\lambda})$. The general solution of this equation is

$$\hat{v}^2 = \hat{v}_r^2 - 2\alpha\left(\hat{\lambda} - \hat{\lambda}_r + \eta\kappa\ln\left(\frac{\hat{\lambda} - \eta\kappa}{\hat{\lambda}_r - \eta\kappa}\right)\right) \tag{5.2}$$

where $(\hat{v}_r, \hat{\lambda}_r)$ denotes any fixed reference point.

For $\kappa > 0$, (5.1) has a unique critical point at (0, 0). The eigenvalues and corresponding eigenvectors are

$$\rho_\pm = \pm\nu(\alpha\eta\kappa)^{1/2}, \qquad \mathbf{V}_\pm = \begin{pmatrix} 1 \\ \pm(\eta\kappa/\alpha)^{1/2} \end{pmatrix}$$

Thus the critical point is a saddle.

The phase plane structure for (5.1) is shown in Fig. 5.1. Note that the horizontal line $\hat{\lambda} = \eta\kappa$ is a phase curve for (5.1), as well as a horizontal asymptote for all nearby phase curves. This line corresponds to the $\lambda = 1$ line of the original vector field (3.7) (although the location of the $\lambda = 1$ line is perturbed slightly from $\eta\kappa$). The region above the $\hat{\lambda} = \eta\kappa$ line is non-physical.

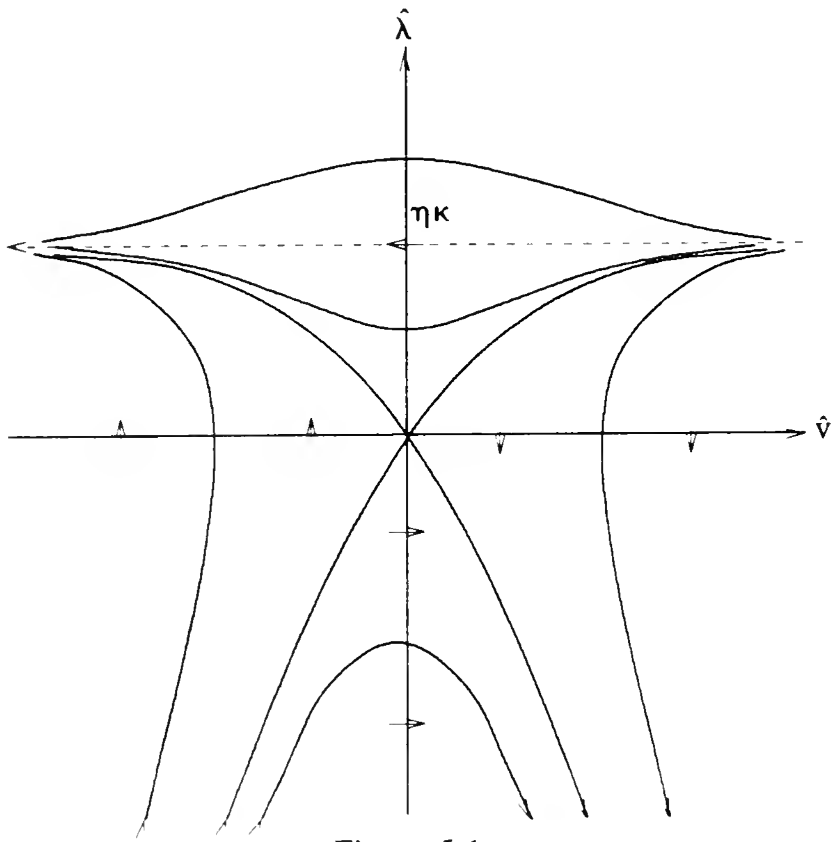

Figure 5.1

We may use (5.2) to eliminate $\hat{v}$ from the $\hat{\lambda}_y$ equation in (5.1), obtaining

$$\hat{\lambda}_y = \mathrm{sgn}(\hat{v}_r)\nu(\hat{\lambda} - \eta\kappa)\left[\hat{v}_r^2 - 2\alpha\left(\hat{\lambda} - \hat{\lambda}_r + \eta\kappa\ln\left(\frac{\hat{\lambda} - \eta\kappa}{\hat{\lambda}_r - \eta\kappa}\right)\right)\right]^{\frac{1}{2}}.$$

The solution is

$$(5.3) \qquad y = \operatorname{sgn}(\hat{v}_r)\int_{\hat{\lambda}_r}^{\hat{\lambda}} \left[\hat{v}_r^2 - 2\alpha\left(s - \hat{\lambda}_r + \eta\kappa \ln\left(\frac{s - \eta\kappa}{\hat{\lambda}_r - \eta\kappa}\right)\right)\right]^{-\frac{1}{2}} \frac{ds}{\nu(s - \eta\kappa)}.$$

When $\kappa = 0$ the vector field (5.1) becomes

$$(5.4) \qquad \begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \nu\hat{\lambda}\begin{pmatrix} -\alpha \\ \hat{v} \end{pmatrix}.$$

The factor of $\hat{\lambda}$ in the vector field creates a continuum of critical points on the $\hat{\lambda} = 0$ axis. This corresponds to the $\lambda = 1$ stationary manifold that we observed in Section 3 for the plane wave. Let $v_c \geq 0$. The linear part of (5.4) at $(\pm v_c, 0)$ is

$$\mathbf{A} \equiv \nu\begin{pmatrix} 0 & -\alpha \\ 0 & \pm\hat{v}_c \end{pmatrix}.$$

There is a double zero eigenvalue at the origin, and only one eigenspace (the $\hat{v}$ axis). This is a double zero bifurcation point, about which more will be said later. When $\hat{v} = \hat{v}_c$ (resp. $-\hat{v}_c$), there is one zero eigenvalue, with the $\hat{v}$ axis as corresponding eigenspace, and one positive (resp. negative) eigenvalue with a corresponding unstable (resp. stable) separatrix solution. These are simple zero bifurcation points which vanish for κ positive. The $\hat{v}$ axis is the common center manifold for the bifurcation points.

If the factor of $\hat{\lambda}$ is removed from the vector field (5.4) the resulting modified vector field has the same phase curves as the original field except along $\hat{\lambda} = 0$. This modified field has no critical points and thus possesses a continuous structurally stable flow. The phase flow of (5.4) thus consists of

a line of critical points superimposed over a continuous one parameter family of phase curves. Setting $\kappa = 0$ in (5.2) yields the phase curve equation

$$\hat{\lambda} = -(2\alpha)^{-1}\hat{v}^2 - \beta \tag{5.5}$$

where

$$\beta \equiv \hat{v}_r^2 + 2\alpha\hat{\lambda}_r .$$

The phase curves are thus a family of parabolas, symmetric about the $\hat{\lambda}$ axis and concave downward. When $\beta > 0$, the phase curve is a separatrix for the critical points at $(\pm\hat{v}_c, 0)$, where $\hat{v}_c \equiv |\beta|^{1/2}$. When $\beta < 0$, the vertex of the parabola lies below the $\hat{v}$ axis. The $\beta = 0$ phase curve is tangent to the $\hat{v}$ axis at the origin. It is this tangency that produces the second zero eigenvalue at the origin. The phase plane structure of (5.4) is illustrated in Figure 5.2.

Equation (5.5) may be combined with the first of equations (5.4) to obtain an ordinary differential equation

$$\hat{v}_y = 1/2\nu(\hat{v}^2 - \beta)$$

for $\hat{v}(y)$. This equation may be integrated and the solution substituted back into (5.5) to yield $\hat{\lambda}(y)$. The results are

$$\hat{v} = \begin{cases} -\hat{v}_c \tanh(\nu\hat{v}_c y / 2 + \phi_+) & , \ \beta > 0 \ , \ |\hat{v}_r| < \hat{v}_c \\ -\hat{v}_c \coth(\nu\hat{v}_c y / 2 + \phi_+) & , \ \beta > 0 \ , \ |\hat{v}_r| > \hat{v}_c \\ \dfrac{\hat{v}_r}{1 - \hat{v}_r \nu y / 2} & , \ \beta = 0 \\ \hat{v}_c \tan(\nu\hat{v}_c y / 2 + \phi_-) & , \ \beta < 0 \end{cases} \tag{5.6}$$

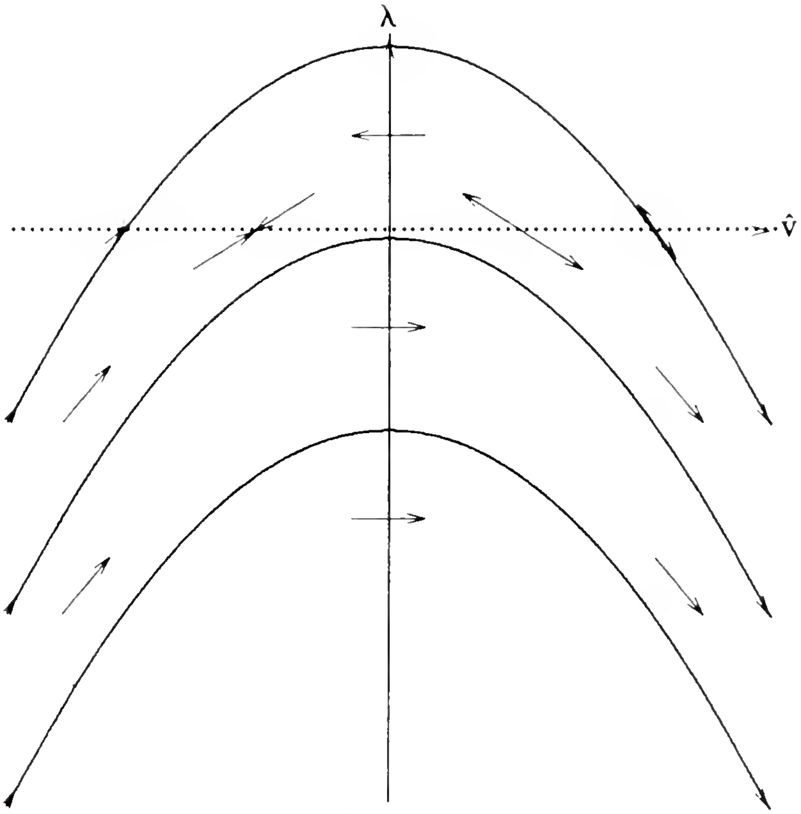

Figure 5.2

$$\hat{\lambda} = \begin{cases} \hat{\lambda}_r \left(\dfrac{\cosh(\phi_-)}{\cosh(\nu \hat{v}_c y \,/\, 2 + \phi_-)} \right)^2 & , \ \beta > 0 \ , \ |\hat{v}_r| < \hat{v}_c \\ \hat{\lambda}_r \left(\dfrac{\sinh(\phi_-)}{\sinh(\nu \hat{v}_c y \,/\, 2 + \phi_-)} \right)^2 & , \ \beta > 0 \ , \ |\hat{v}_r| > \hat{v}_c \\ \dfrac{\hat{\lambda}_r}{(1 - \hat{v}_r \nu y \,/\, 2)^2} & , \ \beta = 0 \\ \hat{\lambda}_r \left(\dfrac{\cos(\phi_-)}{\cos(\nu \hat{v}_c y \,/\, 2 + \phi_-)} \right)^2 & , \ \beta < 0 \end{cases}$$

where

$$\phi_+ = \begin{cases} \tanh^{-1}(-\hat{v}_r / \hat{v}_c), & |\hat{v}_r| < \hat{v}_c \\ \coth^{-1}(-\hat{v}_r / \hat{v}_c), & |\hat{v}_r| > \hat{v}_c \end{cases}$$

$$\phi_- = \tan^{-1}(\hat{v}_r / \hat{v}_c).$$

We end this section with some observations about the double zero bifurcation point at the origin of (5.4). There are two choices of resonant terms for **A** in terms of standard basis vectors. They are

$$\begin{pmatrix} \hat{v}^n \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \hat{v}^n \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ \hat{v}^n \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \hat{v}^{n-1}\hat{\lambda} \end{pmatrix}.$$

Choosing the latter, we find that the only resonant terms in (3.7) at $z = \infty$ are $(0, \hat{\lambda}\hat{v})^T$, and $(0, \hat{\lambda}\hat{v}^2)^T$. Only the second degree term is retained in the proposed normal form (5.4). Note that the resonant terms $(0, \hat{v}^n)^T$ are missing from (3.7) for all n. As a consequence of this degeneracy the bifurcation has infinite codimension: the number of distinct topological equivalence classes which may be obtained by a small perturbation of the plane wave vector field is infinite. This exceptional degeneracy is exhibited as the line of critical points. A nice presentation of the nondegenerate case, in which both second degree resonant terms are present, may be found in Guckenheimer and Holmes [11]. The case of second order degeneracy occurs in models of

chemical reactors [14].

6. The Transonic Critical Point. In this section we apply the Poincaré transformations described in Section 4 to facilitate our study of the phase plane structure of (3.7) in a neighborhood of the sonic bifurcation point. As indicated in Section 3 we will ignore the functional relationship between z and $\dot{z}$ defined by the shooting problem described in Section 3, and consider z and $\dot{z}$ to be independent parameters of the system. It will be seen that this is sufficient to determine the topological structure of the bifurcation point.

In order to carry out the transformations in a way which preserves the correct dependence on κ we employ the standard trick of defining an *augmented* system which consists of (3.7) together with a third equation $\frac{d\kappa}{dy} = 0$. We then expand the augmented system in a Taylor series about the origin $(\hat{v}, \hat{\lambda}, \kappa) = (0, 0, 0)$ and perform the simplifying nonlinear transformations as indicated in Section 4, while allowing only transformations that leave κ invariant.

In what follows we will use Ω to denote a neighborhood of the origin in the $\hat{v}, \hat{\lambda}$ plane. It will be necessary at several points to restrict $(\hat{v}, \hat{\lambda}, \kappa)$ to some cylinder $\Omega_{max} \times [0, \kappa_{max}]$ to obtain a desired result. For notational simplicity we will let κ_{max} and Ω_{max} denote the minimum over all such restrictions. Let $P_{m,n}$ denote the class of real analytic functions on $U \subset R^m$ which are $O(|\mathbf{w}|^n)$ for $\mathbf{w} \in U$, where U is any neighborhood of the origin.

PROPOSITION 6.1 For each

$\bar{p}_i \in \mathcal{P}_{3,2}, \; i \in \{1, 2\},$

$\bar{q}_i \in \mathcal{P}_{2,2}, \; i \in \{1, 2\},$

$a_1, a_2, b_1 > 0,$

$\bar{a}_j \in \mathrm{R}, \; j \in \{3, \cdots, 6\},$

$\bar{b}_k \in \mathrm{R}, \; k \in \{2, 3, 4\},$

there exist $p_i \in \mathcal{P}_{3,2}$, $q_i \in \mathcal{P}_{2,2}$, and $b_k \in \mathrm{R}$, $k \in \{2, 3\}$ such that the system

$$\begin{aligned}
\hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa - \bar{a}_3\hat{v}\hat{\lambda} + \bar{a}_4\kappa\hat{v} + \bar{a}_5\kappa\hat{\lambda} + \bar{a}_6\kappa^2 \\
&\qquad + \kappa\bar{p}_1(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda}\bar{q}_1(\hat{v}, \hat{\lambda}) \\
\hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + \bar{b}_2\kappa\hat{\lambda} + \bar{b}_3\kappa^2 - \bar{b}_4\hat{\lambda}^2 + \kappa\bar{p}_2(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda}\bar{q}_2(\hat{v}, \hat{\lambda}) \\
\kappa_y &= 0
\end{aligned}$$

is smoothly equivalent to the system

$$\begin{aligned}
\hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa + \kappa p_1(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_1(\hat{v}, \hat{\lambda}) \\
\hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + b_2\kappa\hat{\lambda} + b_3\kappa^2 + \kappa p_2(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_2(\hat{v}, \hat{\lambda}) \\
\kappa_y &= 0
\end{aligned} \tag{6.1}$$

at the origin.

If the augmented system is expanded in a Taylor series about the point

$$(v_b, \lambda_b, \kappa_b) \equiv ((v_{CJ}^2 + \mu^2(\dot{z}_0^2 - \dot{z}^2))^{1/2}, 1, 0),$$

it has the form of the first system in Proposition 6.1, providing $0 < v_b < \dot{z}$. When $\dot{z} = \dot{z}_0$, this condition reads $0 < v_{CJ} < \dot{z}_0$. This is satisfied by the plane wave equations (2.7). We may extend this result by continuity to more general $\dot{z}$ providing $\dot{z}$ is restricted to a sufficiently small neighborhood

$(\dot{z}_{\min}, \dot{z}_{\max})$ of $\dot{z}_0$. The values of the coefficients a_i, b_i for our system are

$$a_1 = (\gamma - 1)kq\mathrm{v}_b$$

$$a_2 = \mathrm{v}_b^3(\dot{z} - \mathrm{v}_b)\exp\left(\frac{A\gamma}{\mathrm{v}_b^2}\right)$$

$$b_1 = 2\gamma k\mathrm{v}_b$$

$$b_2 = \exp\left(\frac{A\gamma}{\mathrm{v}_b^2}\right)\left(2\gamma(\dot{z} - \mathrm{v}_b)[\mathrm{v}_b - A(\gamma - 1))]\mathrm{v}_b^3\right)$$

$$b_3 = \frac{\mathrm{v}_b^2(\dot{z} - \mathrm{v}_b)}{(\gamma - 1)kq}\exp\left(\frac{2A\gamma}{\mathrm{v}_b^2}\right) \times$$

$$\left(2\gamma(\dot{z} - \mathrm{v}_b)[\mathrm{v}_b^2 - A(\gamma - 1)] + \mathrm{v}_b^2(4\mathrm{v}_b - 3\dot{z})\right).$$

The coefficients a_1, a_2, and b_1 are positive. The coefficients b_2 and b_3 may be positive, negative, or zero. We point out that the coefficients depend analytically on $\dot{z} \in (\dot{z}_{\min}, \dot{z}_{\max})$. The transformations leading to (6.1) are analytic, uniformly in $\dot{z} \in (\dot{z}_{\min}, \dot{z}_{\max})$, and thus preserve analytic dependence on $\dot{z}$. Thus the p_i, q_i terms must depend analytically on $\dot{z}$. Proposition 6.1 is proven by a rather lengthy application of the transformation methods outlined in Section 4. These calculations were performed with the aid of Macsyma. The calculations are summarized in the appendix. The third equation in (6.1) is no longer needed and will be discarded.

A phase plane analysis yields the next result.

PROPOSITION 6.2 If $p_i = q_i \equiv 0$ in (6.1), the system possesses the unique critical point

$$(6.2) \qquad \hat{\nu}_c = \kappa \frac{a_1 b_3 - a_2 b_2}{a_2 b_1}$$

$$\hat{\lambda}_c = -\kappa \frac{a_2}{a_1}.$$

At the critical point, the linear part

$$A_0(\kappa) = \begin{pmatrix} 0 & -a_1 \\ -\frac{a_2 b_1}{a_1}\kappa & \frac{a_1 b_3}{a_2}\kappa \end{pmatrix}$$

from (6.1) has eigenvalues and eigenvectors given by

$$(6.3) \qquad \rho_\pm = \frac{a_1 b_3 \kappa \pm ((a_1 b_3 \kappa)^2 + 4a_2^3 b_1 \kappa)^{1/2}}{2a_2}$$

$$\mathbf{V}_\pm = \begin{pmatrix} -a_1 \\ \rho_\pm \end{pmatrix}.$$

It is clear from (6.3) and the signs of the coefficients a_2 and b_1 that the critical point in Proposition 6.2 is a saddle for all $b_3 \in \mathrm{R}$, $\dot{z} \in (\dot{z}_{\min}, \dot{z}_{\max})$. The Hartman-Grobman Theorem tells us that all saddle points are topologically equivalent. Consequently for each fixed κ the system (6.1), with p_i and q_i set to zero, is topologically equivalent to the system with b_3 also set to zero. Although the term controlled by b_3 is resonant and cannot be removed by a polynomial change of variables, a more general topological equivalence may indeed remove this term. We thus group the b_3 term with the perturbation terms.

We next prove the existence of a critical point for the system (6.1) for κ sufficiently small, which converges smoothly to the origin as $\kappa \to 0$,

uniformly in $\dot{z}$ (Proposition 6.3). Thus translation of the critical point of (6.1) to the origin constitutes a smooth equivalence transformation. We also show that the saddle point structure persists under the perturbation.

PROPOSITION 6.3 Let $p_i \in \mathcal{P}_{3,2}$, and $q_i \in \mathcal{P}_{2,2}$. For κ_{max} sufficiently small the system (6.1) possesses a critical point $(\hat{v}_c(\kappa, \dot{z}), \hat{\lambda}_c(\kappa, \dot{z})) \in C^\infty([0, \kappa_{max}] \times (\dot{z}_{min}, \dot{z}_{max}), \mathrm{R}^2)$, corresponding through first order in κ to (6.2). There is a neighborhood Ω_{max} of the origin and a κ_{max} such that for $0 < \kappa \leq \kappa_{max}$, this critical point is unique in Ω_{max}, uniformly in $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$.

PROOF: The critical points $(\hat{v}_c, \hat{\lambda}_c)$ of (6.1) are defined by setting the right hand sides of (6.1) to zero. By the Implicit Function Theorem, the first of these equations defines $\hat{\lambda}_c$ as an analytic function of $\hat{v}_c$, $\dot{z}$, and κ in a neighborhood of the point $\hat{v} = \hat{\lambda} = \kappa = 0$, $\dot{z} = \dot{z}_0$, provided $a_1 \neq 0$. It is clear from the definition of a_1 that it is strictly positive in the region of interest. Substituting the expansions

$$\hat{\lambda}_c = \hat{\lambda}^0(\hat{v}) + \kappa\hat{\lambda}^1(\hat{v}) + \cdots$$

$$p_1 = p_1^0(\hat{v}, \hat{\lambda}) + \kappa p_1^1(\hat{v}, \hat{\lambda}) + \cdots$$

in powers of κ into the first equation yields

$$\hat{\lambda}_c = \kappa \frac{p_1^0(\hat{v}, 0) - a_2}{a_1 - q_1(\hat{v}, 0)} + O(\kappa^2),$$

which is well defined for Ω_{max} sufficiently small. Now substitute this result, and a similar expansions for p_2 into the second critical point equation to obtain an implicit equation for $\hat{v}_c(\kappa)$:

$$\kappa\left(\frac{p_1^{1}(\hat{v}_c, 0) - a_2}{a_1 - q_1(\hat{v}_c, 0)} b_1 \hat{v}_c + p_2^{0}(\hat{v}_c, 0)\right) + O(\kappa^2) = 0.$$

The solutions $\kappa = 0$, $\hat{v}_c$ undetermined correspond to the line of critical points discussed in the previous section. Dividing by the common factor of κ, we obtain an equation with the solution $\hat{v}_c = 0$ at $\kappa = 0$. This is the bifurcation point. Observe that since $p_i^{0}(\hat{v}, 0)$, $q_i(\hat{v}, 0) = O(\hat{v}^2)$, the $\kappa = 0$ equation has the form

$$-\hat{v}_c\left(\frac{a_2 b_1}{a_1} + O(\hat{v}_c)\right) = 0.$$

In a sufficiently small neighborhood of the origin, the second factor is non-zero and the $\hat{v}_c = 0$ solution is unique. By the Implicit Function Theorem $\hat{v}_c$ is a single valued smooth function of κ in a neighborhood of $\kappa = 0$ providing $-\frac{a_2 b_1}{a_1}$ is non zero. This follows from (6.1). This solution is unique in some neighborhood Ω_{max} of the origin. Solving for the leading coefficients in the Taylor series expansion of $\hat{v}_c(\kappa)$ we obtain agreement through first order in κ to (6.2). This result can be substituted back into the expansion for $\hat{\lambda}_c(\kappa)$ to obtain first order agreement for $\hat{\lambda}_c$ as well. This completes the proof.

PROPOSITION 6.4 Let $p_i \in \mathcal{P}_{3,2}$ and $q_i \in \mathcal{P}_{2,2}$. For κ_{max} sufficiently small the critical point in Proposition 6.3 is a saddle point.

PROOF: We showed in the previous proposition that the critical point $(\hat{v}_c, \hat{\lambda}_c)(\kappa)$ is $O(\kappa)$ to lowest order. The perturbation terms p_i are second order in $\hat{v}$, $\hat{\lambda}$, and κ, so when evaluated at the critical point they are $O(\kappa^2)$. The linear part of the perturbed system at the critical point thus has the form

Denoting by (v_r, λ_r) any fixed reference point, we have

$$(3.10) \qquad v + \frac{v_b^2 - 2q\mu^2(1 - \lambda)}{v} = v_r + \frac{v_b^2 - 2q\mu^2(1 - \lambda_r)}{v_r} .$$

The choice $v_r = v_b$, $\lambda = 1$ yields the separatrix solution

$$(3.11) \qquad (v - v_b)^2 + 2q\mu^2(\lambda - 1) = 0$$

for the plane wave. This result could also have been obtained by solving (2.7) for $v(\lambda)$.

Denote by $v = g(\lambda, z, \dot{z})$ the solution of the shooting problem for (3.9) connecting the ambient state to a sonic critical point. Evaluating g at the critical point, one obtains an implicit ordinary differential equation

$$v_c(z, \dot{z}) = g(\lambda_c(z, \dot{z}), z, \dot{z})$$

for the radius $z(t)$ and the wave speed $\dot{z}(t)$.

4. Statement of Main Results. The main result of this paper, taken in conjunction with Bukiet [5, 6] is that the velocity of the expanding detonation is equal to the plane wave velocity plus a correction which is to lowest order linear in the shock curvature $\kappa \equiv 1/z$. Moreover the constant of proportionality is determined by the separatrix solution of (3.7). This statement follows on a numerical level from our equations together with their numerical solution and the comparison to the numerical solution of the full Euler equations by Bukiet. One consequence of this result is that standard methods of computation of detonation waves [13] which use the experimental values of the planar detonation velocity can be improved in accuracy by these

corrections. Moreover since the correction can be computed from the chemistry, we believe that the correction can be predicted from some phenomenological equation of state and rate law, at least after the latter have been recalibrated to reflect the new requirement that they reproduce both planar speeds and leading order curvature corrections. Such a predictive capability would minimize the amount of experimental calibration necessary to use this new theory in numerical computations.

We begin our analysis of the model (3.7) with the statement of our main theorem. We will see in subsequent sections that the linear part of the vector field at the plane wave sonic critical point is singular, and is consequently a bifurcation point for the system. Theorem 4.1 identifies the effect of curvature in (3.7) as a perturbation of the sonic bifurcation point.

THEOREM 4.1 Assume that there is a radius z_1 such that $T_c < T$ whenever $0 < x$ and $z_1 < z$. Then there is a $\kappa_{max} > 0$ and a neighborhood $(\dot{z}_{min}, \dot{z}_{max})$ of $\dot{z}_0$ such that

i) The sonic bifurcation point $v_b \equiv (v_{CJ}^2 + \mu^2(\dot{z}^2 - \dot{z}_0^2))^{1/2}$, $\lambda_b = 1$ in the vector field (3.7) bifurcates into a saddle point as $\kappa \equiv \dfrac{1}{z}$ is increased from zero, for all $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$.

ii) For κ_{max} sufficiently small, and $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$, the saddle point in i) is the unique sonic critical point of (3.7).

iii) The location of the saddle point in the phase plane is a C^∞ function of $\kappa \in (0, \kappa_{max})$ and $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$.

iv) The restriction of the vector field (3.3) to the stable separatrix of the saddle point is continuous, uniformly in $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$.

$$(6.5)\qquad \begin{pmatrix}\hat{v}\\ \hat{\lambda}\end{pmatrix}_{\nu} = \left(A_0(\kappa) + \kappa^2 B(\kappa)\right)\cdot\begin{pmatrix}\hat{v}\\ \hat{\lambda}\end{pmatrix} \equiv A(\kappa)\cdot\begin{pmatrix}\hat{v}\\ \hat{\lambda}\end{pmatrix}$$

where $B(\kappa)$ is a smooth matrix function. The critical point is a saddle if and only if the determinant of $A(\kappa)$ is negative. This determinant has the form $-\kappa a_2 b_1 + O(\kappa^2)$, which is negative for κ small but positive, proving the proposition.

Denote the critical point of (6.1) by $\mathbf{w}(\kappa)$. Then the translation $\mathbf{T}$ defined by $\mathbf{T}(\mathbf{w}, \kappa) \equiv \mathbf{w} + \mathbf{w}(\kappa)$ is a global diffeomorphism mapping the origin onto the critical point of (6.1). From Propositions 6.2 and 6.3 we know that $\mathbf{T}$ depends smoothly on κ. If we denote the vector field (6.1) by $\mathbf{G}(\mathbf{w}, \kappa)$, and define the translated field $\mathbf{g}(\mathbf{w}, \kappa) \equiv \mathbf{G}(\mathbf{T}(\mathbf{w}, \kappa), \kappa)$, we have the commutation relations

$$\mathbf{T}(\phi_\nu^{\mathbf{G}}(\mathbf{w}), \kappa) = \phi_\nu^{\mathbf{g}}(\mathbf{T}(\mathbf{w}, \kappa))$$

between the translated and untranslated phase flows. We thus have the following result.

THEOREM 6.5 After translation to the unique critical point in Ω_{max} given by Proposition 6.3, the vector field (6.1) has the form

$$(6.6)\qquad \begin{pmatrix}\hat{v}\\ \hat{\lambda}\end{pmatrix}_{\nu} = \nu\begin{pmatrix} -\alpha\hat{\lambda} + \kappa p_1(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_1(\hat{v}, \hat{\lambda}) \\ \hat{v}\hat{\lambda} - \eta\kappa\hat{v} - \omega\kappa\hat{\lambda} + \kappa p_2(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_2(\hat{v}, \hat{\lambda})\end{pmatrix},$$

where

$$\nu = b_1, \qquad \alpha = a_1/b_1,$$
$$\eta = a_2/a_1, \qquad \omega = \alpha b_3/b_1.$$

$q_i \in P_{2,2}$, and $p_i(\mathbf{w}, \kappa) = O(\kappa|\mathbf{w}| + \mathbf{w}^2)$.

The coefficients α, ν and η are identical to those introduced in Section 6. The vector field (6.6) will be denoted by $\mathbf{g}(\mathbf{w}, \kappa)$.

PROOF: The coefficients are obtained by explicit calculation from the critical point derived in Proposition 6.3. The conditions on p_1 and p_2 follow from the requirement that the critical point of (6.6) be at the origin for all $\kappa \in [0, \kappa_{max}]$. This proves the theorem.

7. Proof of Theorem 4.1 In Section 6 we have identified the phase plane structure of (3.7) in a neighborhood of the plane wave sonic bifurcation point. In the present section we prove Theorem 4.1. In the course of the proof we will exhibit the topological structure of the solutions for small κ in the physically relevant region of the phase plane.

Define

(7.1) $$\psi(v, \lambda, \kappa, z) \equiv q(\gamma - 1)k(1 - \lambda) - \kappa(z - v)c^2 \exp\left(\frac{A\gamma}{c^2}\right),$$

where c^2 is given by (3.8). The vector field (3.7) then becomes

(7.2) $$\begin{aligned} v_y &= v\psi(v, \lambda, \kappa, z) \\ \lambda_y &= k(1 - \lambda)(c^2 - v^2). \end{aligned}$$

As always, we assume that q, c_a, k, z, κ and A are positive and that $\gamma > 1$. The function ψ is analytic providing we avoid a vacuum state $c^2 = 0$. The set of possible vacuum states in the v, λ plane is determined by setting $c^2 = 0$ in Bernoulli's Law (3.8), and will be denoted the *vacuum locus*. The result is

$$(6.5) \qquad \begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}_y = \left(A_0(\kappa) + \kappa^2 B(\kappa)\right) \cdot \begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix} \equiv A(\kappa) \cdot \begin{pmatrix} \hat{v} \\ \hat{\lambda} \end{pmatrix}$$

where $B(\kappa)$ is a smooth matrix function. The critical point is a saddle if and only if the determinant of $A(\kappa)$ is negative. This determinant has the form $-\kappa a_2 b_1 + O(\kappa^2)$, which is negative for κ small but positive, proving the proposition.

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where

$$\nu = b_1, \qquad \alpha = a_1/b_1,$$
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$q_i \in \mathcal{P}_{2,2}$, and $p_i(\mathbf{w}, \kappa) = O(\kappa|\mathbf{w}| + \mathbf{w}^2)$.

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Define

$$(7.1) \qquad \psi(v, \lambda, \kappa, z) \equiv q(\gamma - 1)k(1 - \lambda) - \kappa(z - v)c^2 \exp\left(\frac{A\gamma}{c^2}\right),$$

where c^2 is given by (3.8). The vector field (3.7) then becomes

$$(7.2) \qquad \begin{aligned} v_y &= v\psi(v, \lambda, \kappa, z) \\ \lambda_y &= k(1 - \lambda)(c^2 - v^2). \end{aligned}$$

As always, we assume that q, c_a, k, z, κ and A are positive and that $\gamma > 1$. The function ψ is analytic providing we avoid a vacuum state $c^2 = 0$. The set of possible vacuum states in the v, λ plane is determined by setting $c^2 = 0$ in Bernoulli's Law (3.8), and will be denoted the *vacuum locus*. The result is

$$2q\mu^2(1 - \lambda) = -\mu^2 v^2 + v_b^2.$$

The equation

$$2q\mu^2(1 - \lambda) = -v^2 + v_b^2$$

defining the sonic locus $v^2 = c^2$ is also a parabola. The sonic and vacuum loci possess the same vertex $(0, 1 - v_b^2/2q\mu^2)$ and axis of symmetry (λ axis). From (3.5) we find that $v_{CJ} > 2q\mu^2$, consequently $1 - v_b^2/2q\mu^2 < 0$ for $\dot{z}$ in a sufficiently small neighborhood $(\dot{z}_{\min}, \dot{z}_{\max})$ of $\dot{z}_0$, so that the vertex lies below the v axis. The right intercept with the $\lambda = 1$ line for the sonic locus is v_b. Note that

$$\dot{z}^2 - v_b^2 = \dot{z}_0^2 - v_{CJ}^2 + \frac{2}{\gamma + 1}(\dot{z}^2 - \dot{z}_0^2).$$

From (3.5) we have

$$\dot{z}_0^2 - v_{CJ}^2 = \frac{2}{\gamma + 1}(\dot{z}_0^2 - c_a^2 - (\gamma - 1)q)$$

$$= \frac{2}{\gamma + 1}(\gamma(\gamma - 1)q + (((\gamma^2 - 1)qc_a^{-2} + 1)^2 - 1)^{1/2}c_a^2) > 0,$$

so that $\dot{z} - v_b > 0$ for $\dot{z}$ sufficiently close to $\dot{z}_0$. This means that the right branch of the sonic locus up to $\lambda = 1$ lies to the left of the line $v = \dot{z}$. The vacuum locus is broader than the sonic locus. In fact, from (3.2) we see that $c^2 > 0$ is equivalent to

$$q(\gamma - 1)\lambda + c_a^2 + \frac{\gamma - 1}{2}(\dot{z}^2 - v^2) > 0.$$

This inequality holds in a neighborhood of the rectangle

$$\Sigma(\dot{z}) = \{(v, \lambda) : 0 \leq v \leq \dot{z}, 0 \leq \lambda \leq 1\}.$$

The function ψ is thus analytic on a neighborhood of $\Sigma(\dot{z})$, and we will now restrict our attention to this compact set.

The flow velocity $v = \dot{z}$ of corresponds to a stagnation point $u = 0$ in the original Newtonian frame of reference. The critical point $(\dot{z}, 1)$ is an artifact of the stagnation point that occurs at the center of the spherical detonation.

Our next objective is to identify all of the critical points of (7.2) in $\Sigma(\dot{z})$. This is accomplished by considering the zeros of each of the factors of (7.2), and cataloging the relevant common solutions. When $\kappa = 0$, the $\lambda = 1$ line becomes a manifold of critical points, each possessing at least one zero eigenvalue. There are no other critical points at $\kappa = 0$. As pointed out in Section 3, the λ axis and the line $\lambda = 1$ are fixed phase curves which meet at a fixed critical point $(v, \lambda) = (0, 1)$, which is the unique common zero of the v and $1 - \lambda$ factors of (7.2). The identity $\psi(\dot{z}, 1, \kappa, \dot{z}) \equiv 0$ results in a fixed critical point at $(\dot{z}, 1)$. For $\kappa > 0$, this is the unique common zero of the ψ and $(1 - \lambda)$ factors. For positive κ, the critical point $(0, 1)$ is a sink (two negative eigenvalues), and the critical point $(\dot{z}, 1)$ is a source. The factors v and $c^2 - v^2$ have no common zeros in $\Sigma(\dot{z})$. As shown in the previous section, there is a saddle point on the sonic locus for positive κ, which converges smoothly to the sonic bifurcation point $(v_b, 1)$ as $\kappa \to 0$. The saddle point is at an intersection of the solution of $\psi = 0$ with the sonic locus. We claim that for κ sufficiently small, these three critical points are the only critical points of (7.2) in $\Sigma(\dot{z})$. It is clear that any additional critical point must result from another common solution of $\psi = 0$ and $c^2 = v^2$. At $\kappa = 0$, the equation $\psi(v, \lambda, \kappa, \dot{z}) = 0$ has the unique solution $\lambda = 1$, independent of v

and $\dot{z}$. More precisely, for each $\dot{z}_1 \in (\dot{z}_{\min}, \dot{z}_{\max})$, and each $v_1 \in [0, \dot{z}_1]$, we have a solution $\psi(v_1, 1, 0, \dot{z}_1) = 0$. Since ψ is smooth on $\Sigma(\dot{z})$, we may apply the Implicit Function Theorem at each of these solutions to show the existence of a unique smooth local solution $\lambda = f(v, \kappa, \dot{z})$ in a neighborhood of $(v_1, 0, \dot{z}_1)$ satisfying $f(v_1, 0, \dot{z}_1) = 1$. We have from (7.1),

$$\frac{d\psi}{d\lambda}(v_1, 1, 0, \dot{z}_1) = -q(\gamma - 1)k,$$

which is strictly nonzero, independent of (and therefore uniformly in) v_1, $\dot{z}_1$. This shows that the solution for $\kappa > 0$ is a smooth curve, deformed slightly from $\lambda = 1$. This is the unique solution in a neighborhood

$$B_\delta \equiv \{(v, \lambda) : 0 \le v \le \dot{z}, 1 - \delta < \lambda < 1 + \delta\}$$

of the $\lambda = 1$ boundary. Define $\Sigma_\delta \equiv \Sigma(\dot{z}) \setminus B_\delta$. We claim that for κ sufficiently small, no additional branches of $\psi = 0$ are created in $\Sigma(\dot{z})$. Specifically, we shall show that for every $\delta > 0$ there is a $\kappa_{\max} > 0$ such that ψ is non-vanishing on the compact set $K(\delta, \kappa_{\max})$ defined by

$$K(\delta, \kappa_{\max}) \equiv \Sigma_\delta \times [0, \kappa_{\max}] \times [\dot{z}_{\min}, \dot{z}_{\max}].$$

Fix $\delta > 0$. Since $\lambda = 1$ is the unique solution of $\psi = 0$ at $\kappa = 0$, ψ is nonzero on $K(\delta, 0))$. By the continuity of ψ, there is a neighborhood N_δ of $K(\delta, 0)$ such that ψ is non-zero on N_δ. Since $K(\delta, 0)$ is compact we can choose N_δ to be bounded. Let

$$\kappa_{\max} \equiv 1/2 \inf_{p \in \partial N_\delta,\, q \in K(\delta, 0)} |p - q|.$$

Because $K(\delta, 0)$ is compact and N_δ is open the infinum is nonzero. Then $K(\delta, \kappa_{\max}) \subset N_\delta$, so that ψ is non-vanishing on $K(\delta, \kappa_{\max})$.

Differentiating (7.1) implicitly with respect to κ we find

$$f_\kappa(\text{v}, 0, \dot{z}) = -\frac{(\dot{z} - \text{v})c^2}{q(\gamma - 1)k} \exp\left(\frac{A\gamma}{c^2}\right),$$

which is negative for $\text{v} < \dot{z}$ and positive for $\text{v} > \dot{z}$, with a simple zero at $\text{v} = \dot{z}$. Thus the $\psi = 0$ curve enters $\Sigma(\dot{z})$ through the $\text{v} = 0$ boundary with $\lambda < 1$, and exits through the critical point $\text{v} = \dot{z}$, $\lambda = 1$. Note that any two points in $\Sigma(\dot{z})$ on the $\psi = 0$ curve are connected by that curve in $\Sigma(\dot{z})$. The same is true for two points on the sonic locus. For κ sufficiently small, the $\psi = 0$ curve must cross the sonic locus at least once. Now the slope f_v may be bounded in an arbitrarily small neighborhood of zero by restricting κ. The slope of the sonic locus in $\Sigma(\dot{z})$ is bounded away from zero, since the vertex lies on the negative λ axis, so that the slopes of the $\psi = 0$ curve and the sonic locus in $\Sigma(\dot{z})$ are in disjoint intervals, uniformly in κ and $\dot{z}$, for sufficiently small κ. Thus the intersection of the two curves is unique in $\Sigma(\dot{z})$. This is just an example of the general result that transversality of two smooth curves in the plane is stable under perturbations of the curves.

Note that $\nabla\psi = (0, -q(\gamma - 1)k) + O(\kappa)$, so that $\psi > 0$ below the $\psi = 0$ curve. Now define the sectors $\Sigma_{\pm\,\pm}$ as follows.

$$\Sigma_{++} = \{(\text{v}, \lambda) \in \Sigma(\dot{z}) : \psi > 0,\ c^2 - \text{v}^2 > 0\}$$

$$\Sigma_{+-} = \{(\text{v}, \lambda) \in \Sigma(\dot{z}) : \psi > 0,\ c^2 - \text{v}^2 < 0\}$$

$$\Sigma_{-+} = \{(\text{v}, \lambda) \in \Sigma(\dot{z}) : \psi < 0,\ c^2 - \text{v}^2 > 0\}$$

$$\Sigma_{--} = \{(\text{v}, \lambda) \in \Sigma(\dot{z}) : \psi < 0,\ c^2 - \text{v}^2 < 0\}$$

The signs of the components of the vector field (7.2) at some point (v, λ)

(and therefore the quadrant into which the vector points) are determined by the sector $\Sigma_{\pm\pm}$ containing (v, λ), except possibly at a critical point or at a boundary. We have seen that the only critical point in the closure $\bar{\Sigma}_{--}$ of Σ_{--} is the saddle point at $\psi = c^2 - v^2 = 0$. A phase curve which crosses $\psi = 0$ away from a critical point must do so vertically, i.e. $d\lambda/dv = \pm\infty$. Likewise, a phase curve which crosses the sonic locus at a non-critical point must have zero slope. Further, the sign of the slope of a phase curve which crosses either the $\psi = 0$ curve or the sonic locus at a non-critical point must change, since exactly one of the components of (7.2) reverse sign. Our final observation is that a non-critical intersection of a phase curve with $\psi = 0$ or with the sonic locus in $\Sigma(\dot{z})$ is transverse, so that the slope of the phase curve changes sign at the intersection. A non-transverse intersection with the $\psi = 0$ curve can only occur at a point where the curve has a vertical tangent. We have shown that the slope of the $\psi = 0$ curve may be bounded in an arbitrarily small neighborhood of 0 by restricting κ_{max}, so no non-transverse intersections are possible. Likewise, non-transverse intersections with the sonic locus may be excluded because the sonic locus is never horizontal in $\Sigma(\dot{z})$. These considerations make possible a classification of the non-critical crossings with the $\psi = 0$ curve and with the sonic locus. Denote by (v_i, λ_i) the coordinates of a non-critical crossing. The possible crossings of a phase curve with the $\psi = 0$ curve are

case	$\lambda < \lambda_i$	$\lambda > \lambda_i$
$a1$	$v_y < 0$, $\lambda_y > 0$	$v_y > 0$, $\lambda_y > 0$
$b1$	$v_y > 0$, $\lambda_y > 0$	$v_y < 0$, $\lambda_y > 0$

$c1$	$v_v > 0, \lambda_v < 0$	$v_v < 0, \lambda_v < 0$
$d1$	$v_v < 0, \lambda_v < 0$	$v_v > 0, \lambda_v < 0.$

The possible non-critical intersections with the sonic locus are

case	$v < v_i$	$v > v_i$
$a2$	$v_v > 0, \lambda_v > 0$	$v_v > 0, \lambda_v < 0$
$b2$	$v_v > 0, \lambda_v < 0$	$v_v > 0, \lambda_v > 0$
$c2$	$v_v < 0, \lambda_v < 0$	$v_v < 0, \lambda_v > 0$
$d2$	$v_v < 0, \lambda_v > 0$	$v_v < 0, \lambda_v < 0.$

Figure 7.1 illustrates cases $a1-d1$.

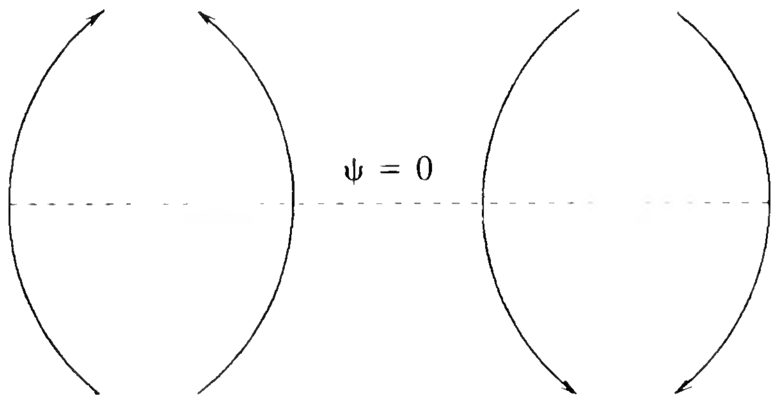

Figure 7.1

We will show that the stable separatrix of the saddle point has positive slope and intersects the $\lambda = 0$ boundary of $\Sigma(z)$. We will accomplish this in two steps. First we show that the stable eigenspace enters the Σ_{--} sector at the saddle point, and that the unstable eigenspace does not. Then we prove

that there are no subsequent intersections of the separatrix with the $\psi = 0$ curve or with the sonic locus. Since Σ_{--} contains no other critical points, and since the $v = 0$ boundary of Σ_{--} is a phase curve, the separatrix must then exit through the $\lambda = 0$ boundary.

We claim that for $\kappa > 0$, exactly one separatrix branch enters each of the four sectors. We have shown that the $\psi = 0$ curve is transverse to the sonic locus at the saddle point, so that the sign of the slope of a separatrix which crosses both curves transversely is determined by the sector $\Sigma_{\pm\pm}$ into which the corresponding eigenvector points. Two such eigenvectors which point into the same sector must necessarily have slopes of the same sign. It is therefore sufficient to show that the eigenvectors are not tangent to the sonic locus or to the curve $\psi = 0$, and that one of the eigenvectors has positive slope, and one has negative. Then one separatrix will cross with positive slope from Σ_{--} into Σ_{--}, and the other will cross from Σ_{--} into Σ_{--} with negative slope. Let $\mathbf{f}(v, \lambda)$ denote the vector field (7.2), and ρ, $\mathbf{V}$ an eigenvalue and corresponding eigenvector at the saddle point, so that $D\mathbf{f} \cdot \mathbf{V} = \rho\mathbf{V}$. By Proposition 6.3, the slopes of both eigenvectors converge continuously to zero as $\kappa \to 0$, so we may assume that $\mathbf{V}_1 \neq 0$. Since $\mathbf{V}$ is only defined modulo a nonzero factor, we may set $\mathbf{V}_1 = 1$. Eliminating ρ and solving for $\mathbf{V}_2$, we obtain

$$\mathbf{V}_2 = \frac{\mathbf{f}_{2,2} - \mathbf{f}_{1,1} \pm ((\mathbf{f}_{2,2} - \mathbf{f}_{1,1})^2 - 4\mathbf{f}_{1,2}\mathbf{f}_{2,1})^{1/2}}{2\mathbf{f}_{1,2}}$$

The two solutions correspond to the two independent eigenvectors. A necessary and sufficient condition for $\mathbf{V}_2$ to have both a positive and a negative solution is $\mathbf{f}_{1,2}\mathbf{f}_{2,1} > 0$. For (7.2) we have

$$\mathbf{f}_{1,2} = -vq(\gamma - 1)k + O(\kappa) < 0,$$

and

$$\mathbf{f}_{2,1} = -(1 - \lambda)(\gamma + 1)/2 < 0,$$

so this condition is satisfied. For $\mathbf{V}$ to be tangent to the sonic locus we must have $\mathbf{V} \cdot \nabla\mathbf{f}_2 = 0$. After eliminating $\mathbf{V}$ this becomes $\mathbf{f}_{1,2}\mathbf{f}_{2,1} - \mathbf{f}_{1,1}\mathbf{f}_{2,2} = 0$, or $\det(D\mathbf{f}) = 0$, which is impossible at a hyperbolic critical point. The same result holds if the sonic locus is replaced by the $\psi = 0$ curve, so the separatrixes are transverse to both curves. Thus the slopes of the separatrixes near the saddle point are given by the sector $\Sigma_{\pm\pm}$ into which the eigenvector points. This means that for each fixed $\kappa > 0$ there is a neighborhood of the saddle point in which one separatrix branch lies in each of the four sectors.

We next show that the separatrix branch in Σ_{--} connects with the $\lambda = 0$ boundary of $\Sigma(\mp)$, with the slope $d\lambda/dv = \mathbf{f}_2/\mathbf{f}_1$ of the phase curve everywhere positive. We have demonstrated above that the separatrix leaves the saddle point with positive slope. Since $v = 0$ is a phase curve, and since Σ_{--} contains no other critical points, we need only show that the separatrix does not intersect the $\psi = 0$ curve or the sonic locus at a non-critical point in $\Sigma(\mp)$. We will assume that there is a non-critical intersection of the separatrix with $\psi = 0$, or with the sonic locus, and arrive at a contradiction. Proceeding in the negative y direction from the saddle point into Σ_{--}, there is a first intersection. Assume that the first intersection is with $\psi = 0$. (The argument is identical for the sonic locus.) This intersection must have one of the forms $a1-d1$. Since both v_y and λ_y are positive in Σ_{--}, we may eliminate cases $c1$ and $d1$. In case $b1$ the phase curve meets the intersection in the

positive y direction from Σ_{--}. Since we must meet the intersection in the negative y direction, we may eliminate this case as well, which leaves us with case $a1$. The slope of the $\psi = 0$ curve may be bounded in an arbitrarily small neighborhood of 0. Since the phase curve $\lambda = \phi(\text{v})$ must intersect with infinite slope, we see from the definition of case $a1$ that there is a neighborhood of the intersection in which $\phi(\text{v}) > \psi(\text{v})$ on the Σ_{--} side of the $\psi = 0$ curve. However, $\psi = 0$ is the upper boundary of Σ_{--}. More precisely, let (v_1, λ_1) be a point in the interior of Σ_{--}. The vertical line $\text{v} = \text{v}_1$ intersects $\psi = 0$ at a unique point $\lambda_2 = \psi(\text{v}_1)$ where $\lambda_1 < \lambda_2$. Since the intersecting phase curve is smooth, it is locally approximated by its vertical tangent line, and must satisfy $\phi(\text{v}) < \psi(\text{v})$ locally. We have our contradiction.

At this point we will return to a consideration of the vector field (3.3), with the original independent variable x. As pointed out in Section 3, this field possesses the same phase curves as the continuous field (3.7). In particular, there is a separatrix solution which increases monotonically from the v axis to the transonic critical point. Since the transonic critical point is unique, all other transonic solutions must cross the sonic locus at a non-critical point. At such points the velocity gradient v_x is unbounded, and the solution is non-smooth. We show now that for fixed κ sufficiently small, the vector field (3.3) is in fact continuous when restricted to the separatrix, so that a smooth solution $\text{v}(x)$ exists, uniformly in $\dot{z} \in (\dot{z}_{\min}, \dot{z}_{\max})$.

In Proposition 6.3 we proved that the transonic critical point (v_c, λ_c) is a smooth function of κ, and that $1 - \lambda_c = O(\kappa)$, uniformly in $\dot{z}$ in some neighborhood $(\dot{z}_{\min}, \dot{z}_{\max})$ of $\dot{z}_0$. We also have the estimate $1/|\mathbf{V}_2| < O(\kappa^{-1/2})$, uniformly in $\dot{z}$, for the reciprocal of the eigenvector slope. The limit of the ratio

$v_\tau/\lambda_\tau = v_y/\lambda_y$ as one approaches the critical point along a separatrix equals $1/\mathbf{V}_2$. Since λ_τ is continuous, this means that v_τ is continuous along the separatrix if it is bounded there, since the limits from either side must be equal. We have

$$\lim_{(v,\lambda)\to(v_c,\lambda_c)} v_\tau$$

$$= \frac{1}{v_c} k(1-\lambda_c)\exp\left(\frac{-A\gamma}{c_c^2}\right) \lim_{(v,\lambda)\to(v_c,\lambda_c)} \frac{v_\tau}{\lambda_\tau}$$

$$= \frac{1}{v_c} k(1-\lambda_c)\exp\left(\frac{-A\gamma}{c_c^2}\right) \lim_{(v,\lambda)\to(v_c,\lambda_c)} \frac{v_y}{\lambda_y}$$

$$\leq \kappa^{1/2} C(\kappa, \dot{z}),$$

where $C(\kappa, \dot{z})$ is bounded, uniformly in $\kappa \in (0, \kappa_{max}]$, and in $\dot{z} \in (\dot{z}_{min}, \dot{z}_{max})$. Hence v_τ is bounded along the separatrix, and possesses a removable discontinuity at the transonic critical point. This completes the proof of Theorem 4.1.

Denote the x coordinate of the transonic critical point in Theorem 4.1 $x_c(\kappa, \dot{z})$. We expect from physical considerations that $x_c \to \infty$ as $\kappa \to 0$, because the plane wave critical point is at infinity. We may use (3.3b) to write x_c as

$$x_c(\kappa, \dot{z}) = k^{-1} \int_0^{\lambda_c(\kappa, \dot{z})} \frac{v(\lambda, \kappa, \dot{z})\exp(A\gamma/c^2(\lambda, \kappa, \dot{z}))\, d\lambda}{1-\lambda}$$

where $v(\lambda, \kappa, \dot{z})$ and $c(\lambda, \kappa, \dot{z})$ denote v and c along the stable separatrix from $\lambda = 0$ to the critical point $\lambda_c(\kappa, \dot{z})$. In order for the critical point to be correctly modelled in the $z \to \infty$ limit it must remain within the region of

asymptotic validity for the model. This will be so if the shock radius z increases faster than x_c. We now show that this is so.

PROPOSITION 7.1

$$\lim_{\kappa \to 0} \kappa x_c(\kappa, z) = 0.$$

PROOF: We have

$$x_c(\kappa, z) \leq \sup_{0 \leq \lambda \leq \lambda_c} |k^{-1}v(\lambda, \kappa, z)\exp(A\gamma/c^2(\lambda, \kappa, z))| \int_0^{\lambda_c(\kappa, z)} \frac{d\lambda}{1 - \lambda}$$

$$\leq \sup_{0 \leq \lambda \leq 1} |k^{-1}v(\lambda, \kappa, z)\exp(A\gamma/c^2(\lambda, \kappa, z))|(-\log(1 - \lambda_c(\kappa, z)).$$

In the proof for Theorem 4.1 we found that the separatrix solution is bounded, uniformly in κ and z, and bounded away from the vacuum locus $c = 0$, so the supremum is bounded. Since λ_c is a smooth function of κ and z,we have

$$\lim_{\kappa \to 0} \kappa \log(1 - \lambda_c(\kappa, z)) = 0.$$

This proves the proposition.

8. Conclusion. In Theorem 4.1 we show that the vector field (3.7) possesses the familiar saddle point structure characteristic of weak detonations. The story does not end here, however. The degenerate double zero eigenvalue bifurcation point present in the velocity-reaction progress plane for the ZND plane wave is infinite codimensional. That is, an infinite number of parameters are required to produce all possible topological

equivalence classes which can be obtained by a smooth perturbation of the system. Only one bifurcation parameter ($1/z$) arises in the present analysis. It may be that a generalization of the model will reveal additional bifurcation parameters which will produce a more complete unfolding of the bifurcation. Also, additional bifurcations of the sonic critical point may alter the phase plane structure at values of the shock radius z which are large compared to the reaction zone width but small with respect to the rather stringent limits imposed in Theorem 4.1. A study of these bifurcations could shed light on phenomena such as detonation failure and reinitiation.

Aknowledgements The results of Sections 1-3 were circulated in manuscript form in the summer of 1985 at the Los Alamos National Laboratory and were presented in a public lecture there in January 1986. We thank J. Bdzil for comments on these occasions. The author also wishes to thank B. Bukiet for his efforts in the numerical validation and implementation of the model. A critical reading of a preliminary version of the manuscript by W. Fickett, R. Menikoff, B. Plohr, and D. H. Sharp provided valuable insights and corrections. The author is indebted to J. Glimm for his continual guidance.

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Appendix. We summarize here the construction of the Poincaré transformations leading to equations (6.1). The augmented system may be written in the form

$$(A.1)\qquad \begin{aligned} \hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa - \bar{a}_3\hat{v}\hat{\lambda} + \bar{a}_4\kappa\hat{v} + \bar{a}_5\kappa\hat{\lambda} + \bar{a}_6\kappa^2 \\ &\qquad + \kappa\bar{p}_1(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda}\bar{q}_1(\hat{v}, \hat{\lambda}) \\ \hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + \bar{b}_2\kappa\hat{\lambda} + \bar{b}_3\kappa^2 - \bar{b}_4\hat{\lambda}^2 + \kappa\bar{p}_2(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda}\bar{q}_2(\hat{v}, \hat{\lambda}) \\ \kappa_y &= 0 \end{aligned}$$

where $\hat{v} \equiv v - v_b$, $\hat{\lambda} \equiv \lambda - 1$, $p_i \in \mathcal{P}_{3,2}$ and $q_i \in \mathcal{P}_{2,2}$. For the augmented system, the coefficients $\bar{b}_2$, $\bar{b}_3$, and $\bar{b}_3$ are zero, as are the terms $\hat{\lambda}\bar{q}_1$ and $\kappa\bar{p}_2$, but we will do the more general case. The matrix of the linear part of $(A.1)$ is

$$(A.2)\qquad \mathbf{A} \equiv \begin{pmatrix} 0 & -a_1 & -a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We seek smooth transformations of the phase space eliminating second order terms of the vector field while leaving the third variable (here, κ) invariant. The induced linear operator $\mathbf{L_A}$ on $\mathcal{J}_{3,2}$ may be computed from (4.1) by its action on the standard basis. The results are

$(A.3)$

$\mathbf{h}$	$\mathbf{L_A} \cdot \mathbf{h}$	$\mathbf{h}$	$\mathbf{L_A} \cdot \mathbf{h}$
$\begin{pmatrix} \hat{v}^2 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -2\hat{v}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \hat{v}\hat{\lambda} \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -\hat{\lambda}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix}$

$$
\begin{array}{ll|ll}
\begin{pmatrix} \hat{v}\kappa \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} -\kappa(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \hat{\lambda}^2 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \hat{\lambda}\kappa \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \kappa^2 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ \hat{v}^2 \\ 0 \end{pmatrix} & \begin{pmatrix} a_1\hat{v}^2 \\ -2\hat{v}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \hat{v}\hat{\lambda} \\ 0 \end{pmatrix} & \begin{pmatrix} a_1\hat{v}\hat{\lambda} \\ -\hat{\lambda}(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ \hat{v}\kappa \\ 0 \end{pmatrix} & \begin{pmatrix} a_1\hat{v}\kappa \\ -\kappa(a_1\hat{\lambda} + a_2\kappa) \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \hat{\lambda}^2 \\ 0 \end{pmatrix} & \begin{pmatrix} a_1\hat{\lambda}^2 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ \hat{\lambda}\kappa \\ 0 \end{pmatrix} & \begin{pmatrix} a_1\hat{\lambda}\kappa \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \kappa^2 \\ 0 \end{pmatrix} & \begin{pmatrix} a_1\kappa^2 \\ 0 \\ 0 \end{pmatrix}.
\end{array}
$$

The remaining basis vectors **h** would be used to construct transformations of the bifurcation parameter κ, which are forbidden in Proposition 6.1. We will eliminate the nonresonant terms of $(A.1)$ one term at a time. The order in which the terms are eliminated must be chosen carefully, since a transformation eliminating one term may contribute to another. Five steps are required:

Step 1: Eliminate the $\bar{b}_4$ term using the transformation

$$
\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{\bar{b}_4}{a_1} \begin{pmatrix} 0 \\ \hat{v}\hat{\lambda} \\ 0 \end{pmatrix}.
$$

This transformation contributes to the a_3 and b_2 terms. Specifically $a_3 \to a_3 + a_4$ and $b_2 \to b_2 - a_2 b_4/a_1$. Higher order terms are also created, so

that $(A.1)$ becomes

$$
\begin{aligned}
\hat{v}_y &= -a_1\hat{\lambda} - a_2\kappa - a_3\hat{v}\hat{\lambda} + a_4\kappa\hat{v} + a_5\kappa\hat{\lambda} + a_6\kappa^2 \\
&\qquad + \kappa p_1(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_1(\hat{v}, \hat{\lambda}) \\
\hat{\lambda}_y &= b_1\hat{v}\hat{\lambda} + b_2\kappa\hat{\lambda} + \kappa p_2(\hat{v}, \hat{\lambda}, \kappa) + \hat{\lambda} q_2(\hat{v}, \hat{\lambda}) \\
\kappa_y &= 0
\end{aligned}
$$

where $p_i \in P_{3,2}$ and $q_i \in P_{2,2}$. Since we will perform several transformations, we will hereafter drop the bars on the coefficients and remainders and not bother to adopt distinctive notation for the values at each step of the transformation.

Step 2: Eliminate the a_3 term using the transformation

$$
\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_3}{2a_1}\begin{pmatrix} \hat{v}^2 \\ 0 \\ 0 \end{pmatrix}.
$$

The a_4 term is replaced by $a_4 + a_2 a_3/a_1$.

Step 3: Eliminate the a_4 term using the transformation

$$
\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_4}{a_1}\begin{pmatrix} 0 \\ \hat{v}\kappa \\ 0 \end{pmatrix}.
$$

This transformation contributes to the b_2 and b_3 terms, which are replaced by $b_2 + a_4$ and $b_3 + a_2 a_4/a_1$, respectively.

Step 4: Eliminate the a_5 term using the transformation

$$
\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_5}{a_1}\begin{pmatrix} 0 \\ \kappa\hat{\lambda} \\ 0 \end{pmatrix}.
$$

Step 5: Eliminate the a_6 term using the transformation

$$\begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} \to \begin{pmatrix} \hat{v} \\ \hat{\lambda} \\ \kappa \end{pmatrix} + \frac{a_6}{a_2} \begin{pmatrix} 0 \\ \kappa^2 \\ 0 \end{pmatrix}.$$

We have arrived at (6.1).

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